Problem set 3

Math 218A, Fall 2009

1. Determine with proof whether or not the following functions are weakly differentiable in $\mathbb{R}$, and find the weak derivative if it exists:

$$f(x) = |\sin x|, \qquad g(x) = x \log |x|, \qquad h(x) = x \sin \frac{1}{x}.$$

2. Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ be the open unit ball in $\mathbb{R}^n$ where $n \geq 2$. Prove that

$$u(x) = \log \log \left(1 + \frac{1}{|x|}\right)$$

belongs to $W^{1,n}(\Omega)$, although it is unbounded. (This shows that the imbedding theorem for $1 < p < n$ fails when $p = n$, $p^* = \infty$.)

3. Suppose that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is weakly differentiable and $Du = 0$. Prove that u is a constant.

4. Suppose that $f \in L^2(\mathbb{R}^n)$ with Fourier transform

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int f(x) e^{-i\xi \cdot x}\, dx.$$

(a) Prove that the weak derivative $\partial^\alpha f$ exists and belongs to $L^2(\mathbb{R}^n)$ if and only if $\xi^\alpha \hat{f} \in L^2(\mathbb{R}^n)$.

(b) Prove that $f \in H^k(\mathbb{R}^n)$ for $k \in \mathbb{N}$ if and only if

$$\int \left(1 + |\xi|^2\right)^k \left|\hat{f}(\xi)\right|^2 d\xi < \infty.$$

(c) Prove that if $f \in H^k(\mathbb{R}^n)$ for $k > n/2$, then $f \in C_0(\mathbb{R}^n)$ is a continuous function that decays to zero at infinity. Hint. If $s > n$ then

$$\int_{\mathbb{R}^n} \frac{1}{\left(1 + |\xi|^2\right)^{s/2}}\, d\xi < \infty.$$

You can use standard properties of the Fourier transform on smooth functions and on L^2, such as Parseval's theorem and the Riemann-Lebesgue lemma.