Problem set 4
Math 218A, Fall 2009

1. Consider the Helmholtz equation in $\mathbb{R}^n$

$$-\Delta u + u = f.$$

(a) Use the Riesz representation theorem to prove that there is a unique weak solution $u \in H^1(\mathbb{R}^n)$ for every $f \in H^{-1}(\mathbb{R}^n)$. (Note that $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.)

(b) Prove the same result by use of the Fourier transform.

(c) Is this result true for the Poisson equation $-\Delta u = f$ on $\mathbb{R}^n$? Explain your answer.

2. Prove that any $u \in H_0^2(\Omega)$ satisfies the interpolation inequality

$$\int_\Omega |Du|^2\, dx \le C \left(\int_\Omega u^2\, dx\right)^{1/2} \left(\int_\Omega |D^2 u|^2\, dx\right)^{1/2}.$$

Hint: first use integration by parts to prove this for $u \in C_c^\infty(\Omega)$, then use approximation.

3. Give a precise weak formulation of the biharmonic equation

$$\begin{aligned} \Delta^2 u = f, &\qquad \text{in } \Omega \\ u = \tfrac{\partial u}{\partial n} = 0 &\qquad \text{on } \partial\Omega. \end{aligned}$$

If Ω is a bounded open set, prove that there exists a unique weak solution $u \in H_0^2(\Omega)$ for every $f \in H^{-2}(\Omega)$, where $H^{-2}(\Omega) = H_0^2(\Omega)^*$.

4. Let $\Omega_\alpha \subset \mathbb{R}^2$ be the wedge

$$\Omega_\alpha = \{(r,\theta) \mid 0 < r < 1, \quad 0 < \theta < \alpha\},$$

where (r,θ) are polar coordinates. Use separation of variables to solve the following boundary value problem for Laplace's equation in the wedge:

$$\begin{aligned} \Delta u = 0 &\qquad \text{in } \Omega_\alpha, \\ u = 0 &\qquad \text{when } \theta = 0, \alpha, \\ u = \sin\left(\tfrac{\pi\theta}{\alpha}\right) &\qquad \text{when } r = 1. \end{aligned}$$

Show that $u \in H^1(\Omega_\alpha)$, as required by the general existence theorem, but that $u \notin H^2(\Omega_\alpha)$ when $\alpha > \pi$. (This example shows that global regularity may fail when the boundary is not sufficiently smooth.)