Problem set 1
Math 218B, Spring 2001

1. Prove that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ for $1 \le p < \infty$. (You can assume that $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.)

2. If $n > 1$, show that the unbounded function

$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$$

belongs to $W^{1,n}(B)$, where

$$B = \{x \in \mathbb{R}^n \mid |x| < 1\}.$$

This example shows that Sobolev functions need not be continuous when $p = p^*$.

3. Show that there is no continuous linear map

$$T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^{n-1}),$$

where $1 \le p < \infty$, such that for all $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$(T\phi)(x_1, \dots, x_{n-1}) = \phi(x_1, \dots, x_{n-1}, 0).$$

4. Prove that any $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies the interpolation inequality

$$\int_\Omega |Du|^2\, dx \le C \left(\int_\Omega u^2\, dx\right)^{1/2} \left(\int_\Omega |D^2 u|^2\, dx\right)^{1/2}.$$

Hint: first use integration by parts to prove this for $u \in C_c^\infty(\Omega)$, then use approximation.

5. (Nikodým, 1933) Let $\Omega \subset \mathbb{R}^2$ be the union of the rectangles

$$\begin{aligned}
A_n &= \left\{(x, y) \mid 2^{-(n-1)} - 2^{-(n+1)} < x < 2^{-(n-1)}, \quad 2/3 < y < 1\right\}, \\
B_n &= \left\{(x, y) \mid 2^{-(n-1)} - \epsilon_n < x < 2^{-(n-1)}, \quad 1/3 \le y \le 2/3\right\}, \\
C &= \{(x, y) \mid 0 < x < 1, \quad 0 < y < 1/3\},
\end{aligned}$$

where $0 < \epsilon_n < 2^{-(n+1)}$ and $n \in \mathbb{N}$. Let u be a continuous function on Ω that is equal to α_n on A_n, equal to 0 on C, and linear on B_n. Show that for a suitable choice of the constants α_n and ϵ_n we have $Du \in L^2(\Omega)$ but $u \notin L^2(\Omega)$. (It is possible to prove that if $Du \in L^p(\Omega)$, then $u \in L^p_{\mathrm{loc}}(\Omega)$.)