Problem set 2

Math 218B, Spring 2001

1. Give a counterexample to show that if $\Omega' \subset\subset \Omega$, then $u \in L^1_{\mathrm{loc}}(\Omega)$, and

$$\|D^h u\|_{L^1(\Omega')} \leq C,$$

for all sufficiently small $h > 0$, does not imply that $Du \in L^1(\Omega')$. Where does the proof of the analogous result for $1 < p < \infty$ fail?

2. Consider the following problem for $u \in H^1(\Omega)$:

$$\int_\Omega \left\{ a^{ij} u_{x^i} v_{x^j} + b^i u_{x^i} v + cu \right\} dx = 0 \qquad \text{for all } v \in H^1(\Omega).$$

If the function u is smooth, and the coefficient functions and the domain are also smooth, derive the PDE and boundary conditions that correspond to this weak formulation.

3. Give a precise weak formulation of the *biharmonic* equation

$$\begin{aligned} \Delta^2 u &= f, && \text{in } \Omega \\ u = \tfrac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Prove that there exists a unique weak solution $u \in H^2_0(\Omega)$ for every $f \in L^2(\Omega)$.

4. Suppose that L is an elliptic operator

$$Lu = a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu$$

with constant coefficients a^{ij}, b^i, c, acting on functions $u : \mathbb{R}^n \to \mathbb{R}$. Use the Fourier transform to show that if $u \in L^2(\mathbb{R}^n)$ and $Lu \in L^2(\mathbb{R}^n)$, then $u \in H^2(\mathbb{R}^n)$, and there exists a constant C such that

$$\|u\|_{H^2(\mathbb{R}^n)} \leq C \left(\|Lu\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)} \right).$$

5. Let $\Omega_\alpha \subset \mathbb{R}^2$ be the wedge

$$\Omega_\alpha = \{(r, \theta) \mid 0 < r < 1, \quad 0 < \theta < \alpha\},$$

where (r, θ) are polar coordinates. Use separation of variables to solve the following boundary value problem for Laplace's equation in the wedge:

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega_\alpha, \\ u &= 0 && \text{when } \theta = 0, \alpha, \\ u &= \sin\left(\frac{\pi\theta}{\alpha}\right) && \text{when } r = 1. \end{aligned}$$

Show that $u \in H^1(\Omega_\alpha)$, as required by the general existence theorem, but that $u \notin H^2(\Omega_\alpha)$ when $\alpha > \pi$. (This example shows that global regularity may fail when the boundary is not sufficiently smooth.)