Calculus

Math 21A, Fall 2015

Sample Final Problems: Solutions

1. Compute the derivatives of the following functions. You do not need to simplify your answers.

(a) $\dfrac{2x+5}{\sqrt{x^4+3}}$

(b) $e^{3x+1}\ln(x^2+1)$

(c) $\tan(\cos x+\sin x)$

(d) $\sin^{-1}\left(\sqrt{2x}\right)$

Solution.

- (a)
$$\frac{d}{dx}\frac{2x+5}{\sqrt{x^4+3}} = \frac{2\cdot\sqrt{x^4+3} - \frac{1}{2\sqrt{x^4+3}}\cdot 4x^3\cdot(2x+5)}{x^4+3}$$

- (b)
$$\frac{d}{dx}e^{3x+1}\ln(x^2+1) = e^{3x+1}\cdot 3\cdot\ln(x^2+1) + e^{3x+1}\cdot\frac{1}{x^2+1}\cdot 2x$$

- (c)
$$\frac{d}{dx}\tan(\cos x+\sin x) = \sec^2(\cos x+\sin x)\cdot(-\sin x+\cos x)$$

- (d)
$$\frac{d}{dx}\sin^{-1}\left(\sqrt{2x}\right) = \frac{1}{\sqrt{1-2x}}\cdot\frac{1}{2\sqrt{2x}}\cdot 2$$

2. Evaluate the following limits or say if they do not exist (using any method you want):

(a) $\displaystyle \lim_{x\to 0} \frac{1-\cos(2x)}{xe^x - x}$

(b) $\displaystyle \lim_{x\to 0} \left\{ \sin x \left[\frac{1}{x} - \frac{1}{\sin(2x)} \right] \right\}$

(c) $\displaystyle \lim_{x\to\infty} \left(\ln x \, e^{-x} \right)$

(d) $\displaystyle \lim_{x\to 1} \frac{\sin x}{x}.$

Solution.

- (a) This is an indeterminate limit $(0/0)$, and applying l'Hôpital's rule twice, we get

$$\begin{aligned} \lim_{x\to 0} \frac{1-\cos(2x)}{xe^x - x} &= \lim_{x\to 0} \frac{2\sin(2x)}{xe^x + e^x - 1} \\ &= \lim_{x\to 0} \frac{4\cos(2x)}{xe^x + 2e^x} \\ &= 2. \end{aligned}$$

- (b) Using $\lim_{\theta\to 0} \sin\theta/\theta = 1$, we get

$$\begin{aligned} \lim_{x\to 0} \left\{ \sin x \left[\frac{1}{x} - \frac{1}{\sin(2x)} \right] \right\} &= \lim_{x\to 0} \left\{ \frac{\sin x}{x} - \frac{\sin x}{x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{1}{2} \right\} \\ &= 1 - 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

- (c) Since $\ln x \to \infty$ and $e^x \to \infty$ as $x \to \infty$, we can apply l'Hôpital's rule to get

$$\begin{aligned} \lim_{x\to\infty} \left(\ln x \, e^{-x} \right) &= \lim_{x\to\infty} \frac{\ln x}{e^x} \\ &= \lim_{x\to\infty} \frac{1/x}{e^x} \\ &= 0. \end{aligned}$$

- (d) By the continuity of $\sin x$,

$$\lim_{x\to 1} \frac{\sin x}{x} = \sin 1.$$

3. A 15 ft ladder is leaning against a wall. If the base of the ladder is pushed toward the wall at a speed of 2 ft/sec, at what speed is the top of the ladder moving up the wall when the base of the ladder is 6ft from the wall?

Solution.

- Let $y(t)$ be the height of the top of the ladder and $x(t)$ the distance of the base of the ladder from the wall at time t. Then, by the Pythagorean theorem,
$$x^2 + y^2 = 15^2.$$
If $x = 6$, then $y = \sqrt{15^2 - 6^2} = 3\sqrt{21}$.

- Differentiation with respect to t gives
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0,$$
so
$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

- When $x = 6$, $y = 3\sqrt{21}$, and $dx/dt = -2$, we get
$$\frac{dy}{dt} = \frac{4}{\sqrt{21}} \text{ ft/sec.}$$

4. A pile of sand in the shape of a cone whose radius is twice its height is growing at a rate of 5 cubic meters per second. How fast is its height increasing when the radius is 20 meters? HINT. The volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.

Solution.

- If $r = 2h$, then
$$V = \frac{4}{3}\pi h^3.$$
- Differentiation with respect to time t gives
$$\frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}.$$
- When $r = 20$, $h = 10$, and $dV/dt = 5$, we get
$$\frac{dh}{dt} = \frac{1}{80\pi} \text{ meters/second.}$$

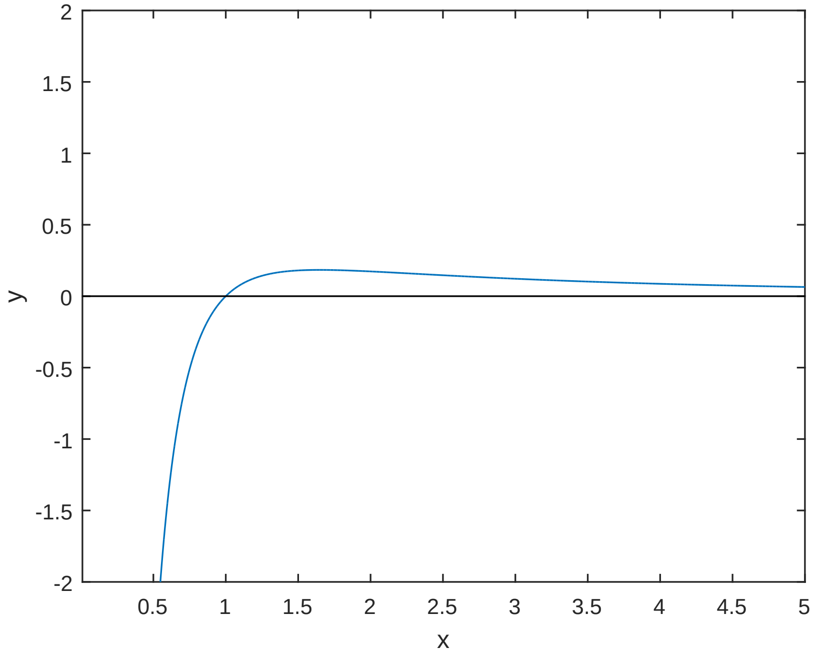
5. State the natural domain of the function

$$y = \frac{\ln x}{x^2}.$$

Sketch the graph, identify where the graph is increasing/decreasing, the local extrema, where the graph is concave up/concave down, and the inflection points.

Solution.

- The domain is $(0, \infty)$, where $x > 0$.
- We have
$$y' = \frac{1 - 2\ln x}{x^3}.$$
- Increasing ($y' > 0$) for $\ln x < 1/2$ or $0 < x < \sqrt{e}$.
- Decreasing ($y' < 0$) for $\ln x > 1/2$ or $x > \sqrt{e}$.
- Global maximum at $(x, y) = (\sqrt{e}, 1/2e)$.
- We have
$$y'' = \frac{6\ln x - 5}{x^6}.$$
- Concave up ($y'' > 0$) for $\ln x > 5/6$ or $x > e^{5/6}$.
- Concave down ($y'' < 0$) for $\ln x < 5/6$ or $0 < x < e^{5/6}$.
- Inflection point at $(x, y) = (e^{5/6}, (5/6)e^{-5/3})$.
- The graph is on the next page ($\sqrt{e} \approx 1.65$, $e^{5/6} \approx 2.30$).


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
x

6. A cricket ball is projected directly upward from the ground with an initial velocity of 112 ft/s. Assuming that the acceleration due to gravity is 32 ft/sec^2, derive an equation for the height $s(t)$ of the ball above the ground after t seconds. When does the ball hit the ground?

Solution.

- Measuring height upwards, the acceleration of the ball is given by

$$\frac{d^2s}{dt^2} = -32.$$

- It follows that

$$\frac{ds}{dt} = -32t + v_0$$

  where v_0 is a constant.

- At $t = 0$, the velocity ds/dt is 112, so $v_0 = 112$.

- It also follows that

$$s = -16t^2 + v_0 t + s_0$$

  where s_0 is a constant.

- At $t = 0$ the height of the ball is 0, so $s_0 = 0$ and

$$s(t) = -16t^2 + 112t.$$

- Since $s = 16t(7 - t)$, the ball hits the ground ($s = 0$) after 7 seconds.

7. A one meter high fence is eight meters in front of a high wall. Find the minimum length of a ladder resting on the fence whose foot is in front of the fence and whose top reaches the wall.

Solution.

- Let y be the height of the top of the ladder at the wall, x the distance of the foot of the ladder from the wall, and a the distance of the foot of the ladder from the fence. Then $x = 8 + a$ and, by similar triangles, $y/x = 1/a$, so
$$y = \frac{x}{x-8}.$$
- Minimizing the length L of the ladder is the same as minimizing the square $S = L^2$ of the length, which is
$$S = x^2 + y^2.$$
- It follows that we want to find the global minimum of
$$S(x) = x^2 + \left(\frac{x}{x-8}\right)^2 \qquad \text{on } 8 \leq x < \infty.$$
- We compute that
$$\begin{aligned} S'(x) &= 2x + 2\left(\frac{x}{x-8}\right)\left[\frac{(x-8)-x}{(x-8)^2}\right] \\ &= 2x - \frac{16x}{(x-8)^3} \\ &= \frac{2x\left[(x-8)^3 - 8\right]}{(x-8)^3} \end{aligned}$$
- There is one critical point with $S'(x) = 0$ and $x > 8$ when $(x-8)^3 = 8$ or $x = 10$. Moreover, $S'(x) < 0$ if $8 \leq x < 10$, so $S(x)$ is decreasing, and $S'(x) > 0$ if $x > 10$, so $S(x)$ is increasing. It follows that $S(x)$ has a global minimum at $x = 10$.
- At $x = 10$, we have $y = 5$ and $S = 125$. The minimum length $\sqrt{S}$ of the ladder is therefore $5\sqrt{5}$ meters.

8. Define a function $f(x)$ with domain $(-\infty, \infty)$ by

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ Ax + B & \text{if } 1 < x < 2 \\ -2x^2 & \text{if } x \ge 2. \end{cases}$$

(a) Determine the constants A and B so that $f(x)$ is continuous everywhere.
(b) Is this function differentiable everywhere?
(c) Sketch the graph $y = f(x)$ in that case and determine the range of f.

Solution.

- (a) To get continuity, we need $Ax+B = x^2$ at $x = 1$ and $Ax+B = -2x^2$ at $x = 2$, or
$$A + B = 1, \qquad 2A + B = -8.$$
The solution of these equations is
$$A = -9, \qquad B = 10.$$

- (b) The function is not differentiable at $x = 1$ or $x = 2$ since the left and right derivatives are different: $f'(1^+) = -9$, $f'(1^-) = 2$; $f'(2^+) = -8$, $f'(2^-) = -9$.

- (c) The range of f is $(-\infty, \infty)$. The graph is omitted.

9. Find the equation of the tangent line to the curve

$$(x-y)^3 = x^2 - y^2 - 2$$

at the point $(2, 1)$. At which point does this tangent line cross the x-axis and at what angle?

Solution.

- Implicit differentiation gives

$$3(x-y)^2\left(1 - \frac{dy}{dx}\right) = 2x - 2y\frac{dy}{dx}.$$

- At the point $(2, 1)$ we get that

$$\frac{dy}{dx} = -1.$$

- The equation of the tangent line is $y - 2 = -(x-1)$ or

$$y = 3 - x.$$

- The tangent line crosses the x-axis at $(x, y) = (3, 0)$. It has slope -1, so the angle is $3\pi/4$.