Calculus
Math 21A, Fall 2015
Midterm 1: Solutions

1. [20%] Say if the following statements are true or false. (For this question only, you don't have to explain your answers).

(a) If $\lim_{x\to c^+} f(x) = 3$ and $\lim_{x\to c^-} f(x) = 3.001$, then $\lim_{x\to c} f(x)$ is close to 3.

(b) If $\lim_{x\to c} f(x)$ exists, then $f(x)$ is continuous at c.

(c) If $f(x)$ is continuous at c, then $\lim_{x\to c} f(x)$ exists.

(d) If $\lim_{x\to 0^+} f(x) = 0$, then $f(x) > 0$ for all $x > 0$ that are sufficiently close to 0.

Solution.

- (a) False. (If the left and right limits are different, then the limit does not exist.)
- (b) False. (The function $f(x)$ need not be defined at c, or its value may be different from the limit.)
- (c) True. (The existence of the limit is part of the definition of continuity.)
- (d) False. (For example, $\lim_{x\to 0^+}(-x) = 0$ but $-x < 0$ for all $x > 0$.)

2. [30%] Evaluate the following limits or say if they do not exist.

(a) $\lim_{x \to 2} \dfrac{2x^2+1}{11-x^3}$

(b) $\lim_{x \to 0} \dfrac{\sqrt{5x+4}-2}{x}$

(c) $\lim_{x \to 0} \dfrac{\sin(1/x)}{x}$

(d) $\lim_{x \to 3^-} \dfrac{x^2-2x-3}{|x-3|}$

Solution.

- (a) Since the limit of the denominator is non-zero, we have

$$\lim_{x \to 2} \frac{2x^2+1}{11-x^3} = \frac{2 \cdot 2^2+1}{11-2^3} = 3.$$

- (b) Using the difference of two squares to simplify the square root, we get

$$\begin{aligned}\lim_{x \to 0} \frac{\sqrt{5x+4}-2}{x} &= \lim_{x \to 0} \frac{\left(\sqrt{5x+4}-2\right)\left(\sqrt{5x+4}+2\right)}{x\left(\sqrt{5x+4}+2\right)} \\ &= \lim_{x \to 0} \frac{5x}{x\left(\sqrt{5x+4}+2\right)} \\ &= \lim_{x \to 0} \frac{5}{\sqrt{5x+4}+2} \\ &= \frac{5}{4}.\end{aligned}$$

- (c) This limit does not exist, since $\sin(1/x)$ oscillates infinitely often between -1 and 1 as $x \to 0$ and the limit of the denominator x is 0.

- (d) If $x < 3$, then $|x - 3| = 3 - x$, so

$$\begin{aligned}\lim_{x \to 3^-} \frac{x^2 - 2x - 3}{|x - 3|} &= \lim_{x \to 3^-} \frac{x^2 - 2x - 3}{3 - x} \\ &= \lim_{x \to 3^-} \frac{(x - 3)(x + 1)}{3 - x} \\ &= -\lim_{x \to 3^-} (x + 1) \\ &= -4\end{aligned}$$

3. [15%] Evaluate the following limits involving infinity.

$$\text{(a)}\ \lim_{x\to 1} \frac{x+1}{(x-1)^2(x-2)}$$
$$\text{(b)}\ \lim_{x\to\infty} \sqrt{\frac{x^2+2x+4}{4x^2+2x+1}}$$
$$\text{(c)}\ \lim_{x\to-\infty} xe^{1/x}$$

Solution.

- (a) Since $(x-1)^2 > 0$, we have

$$\begin{aligned}\lim_{x\to 1} \frac{x+1}{(x-1)^2(x-2)} &= \lim_{x\to 1} \frac{x+1}{x-2} \cdot \lim_{x\to 1} \frac{1}{(x-1)^2} \\ &= -2 \lim_{x\to 1} \frac{1}{(x-1)^2} \\ &= -\infty.\end{aligned}$$

- (b) We have

$$\lim_{x\to\infty} \sqrt{\frac{x^2+2x+4}{4x^2+2x+1}} = \lim_{x\to\infty} \sqrt{\frac{1+2/x+4/x^2}{4+2/x+1/x^2}} = \frac{1}{2}.$$

- (c) We have

$$\lim_{x\to-\infty} e^{1/x} = \lim_{t\to 0^-} e^t = e^0 = 1,$$

so

$$\lim_{x\to-\infty} xe^{1/x} = \lim_{x\to-\infty} x = -\infty.$$

4. [20%] Define a function $f(x)$ for all real numbers except $x = -1, 0, 2$ by

$$f(x) = \begin{cases} 1/x^2 & \text{if } x < -1, \\ -1/x^3 & \text{if } -1 < x < 0, \\ 1/x & \text{if } 0 < x < 2. \\ 1/x^2 & \text{if } x > 2. \end{cases}$$

(a) Evaluate $\lim_{x\to-1^-} f(x)$ and $\lim_{x\to-1^+} f(x)$. Can you choose a value of $f(-1)$ so that $f(x)$ is continuous at $x = -1$; if so, what is $f(-1)$?

(b) Evaluate $\lim_{x\to0^-} f(x)$ and $\lim_{x\to0^+} f(x)$. Can you choose a value of $f(0)$ so that $f(x)$ is continuous at $x = 0$; if so, what is $f(0)$?

(c) Evaluate $\lim_{x\to2^-} f(x)$ and $\lim_{x\to2^+} f(x)$. Can you choose a value of $f(2)$ so that $f(x)$ is continuous at $x = 2$; if so, what is $f(2)$?

Solution.

- (a) We have

$$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} \frac{1}{x^2} = 1,$$
$$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} -\frac{1}{x^3} = 1.$$

  Since the left and right limits are the same, $\lim_{x\to-1} f(x) = 1$ exists and $f(x)$ has a removable discontinuity at $x = -1$. We can make $f(x)$ continuous at -1 by defining $f(-1) = 1$.

- (b) We have

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} -\frac{1}{x^3} = \infty,$$
$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{1}{x} = \infty,$$

  so $f(x)$ has an infinite discontinuity at $x = 0$ and $\lim_{x\to0} f(x)$ does not exist. We cannot make $f(x)$ continuous at 0 by any choice of the value of $f(0)$.

- (c) We have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{1}{x} = \frac{1}{2},$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{1}{x^2} = \frac{1}{4},$$

  so $f(x)$ has a jump discontinuity at $x = 2$ and $\lim_{x \to 2} f(x)$ does not exist. We cannot make $f(x)$ continuous at 2 by any choice of the value of $f(2)$.

5. [15%] (a) Suppose that a function $f(x)$ is defined for all x in an interval about c, except possibly at c itself. Give the precise ϵ-δ definition of

$$\lim_{x \to c} f(x) = L.$$

(b) Use the ϵ-δ definition to prove that

$$\lim_{x \to 2} (3x - 1) = 5.$$

Solution.

- (a) $\lim_{x \to c} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \text{ implies that } |f(x) - L| < \epsilon.$$

- (b) Let $\epsilon > 0$ be given and choose

$$\delta = \frac{\epsilon}{3} > 0.$$

  Then $0 < |x - 2| < \delta$ implies that

$$\begin{aligned} |(3x - 1) - 5| &= 3|x - 2| \\ &< 3\delta \\ &< \epsilon, \end{aligned}$$

  which proves the result.