Calculus
Math 21A, Fall 2015
Midterm 2: Solutions

1. [20%] Compute the derivatives of the following functions. You do not need to simplify your answers.

(a) $7\tan(\pi x) + 3x\tan^{-1}(5x)$ (b) $\dfrac{x + x^3 + x^5}{1 + x^2 + x^4}$

(c) $\sqrt{\sin(x^2)}$ (d) $xe^{-2x}\ln(3x)$

Solution.

- (a)

$$\frac{d}{dx}\left[7\tan(\pi x) + 3x\tan^{-1}(5x)\right]$$
$$= 7\cdot\sec^2(\pi x)\cdot\pi + 3\tan^{-1}(5x) + 3x\cdot\frac{1}{1+(5x)^2}\cdot 5$$

- (b)

$$\frac{d}{dx}\left[\frac{x + x^3 + x^5}{1 + x^2 + x^4}\right]$$
$$= \frac{(1+3x^2+5x^4)(1+x^2+x^4) - (2x+4x^3)(x+x^3+x^5)}{(1+x^2+x^4)^2}$$

Alternatively (and unintentionally), note that

$$\frac{x + x^3 + x^5}{1 + x^2 + x^4} = x \qquad \text{so} \qquad \frac{d}{dx}\left[\frac{x + x^3 + x^5}{1 + x^2 + x^4}\right] = 1.$$

- (c)

$$\frac{d}{dx}\left[\sqrt{\sin(x^2)}\right] = \frac{1}{2\sqrt{\sin(x^2)}}\cdot\cos(x^2)\cdot 2x$$

- (d)

$$\frac{d}{dx}\left[xe^{-2x}\ln(3x)\right] = 1\cdot e^{-2x}\ln(3x) + x(-2e^{-2x})\ln(3x) + xe^{-2x}\cdot\frac{1}{3x}\cdot 3$$

2. [25%] (a) Find an equation for the tangent line to the curve

$$2\left(x^3+y^3\right)=9xy$$

at the point $(x, y) = (1, 2)$.
(b) At what point does the tangent line in (a) intersect the line $y = x$?

Solution.

- (a) Using implicit differentiation, we get

$$6x^2+6y^2\frac{dy}{dx}=9y+9x\frac{dy}{dx}.$$

- Canceling a factor of 3 and evaluating the result at $(x, y) = (1, 2)$, we get

$$2+8\frac{dy}{dx}=6+3\frac{dy}{dx},$$

  which implies that

$$\frac{dy}{dx}=\frac{4}{5}.$$

- The equation of the tangent line at $(x, y) = (1, 2)$ is therefore

$$y-2=\frac{4}{5}(x-1). \tag{1}$$

- (b) When $y = x$ in (1), we get

$$5x-10=4x-4$$

  so $x = 6$, and the lines intersect at $(x, y) = (6, 6)$.

3. [25%] A growing column of ice is in the shape of a perfect cylinder whose height is six times its radius. If the volume of the ice is increasing at a rate of 0.5 inches^3/hour, find the rate at which the height is increasing when the height of the column is 4 inches.

HINT. The volume V of a cylinder with height h and radius r is $V = \pi r^2 h$.

Solution.

- Let $h(t)$ be the height, $r(t)$ the radius, and $V(t)$ the volume of the cylinder at time t. Then $h = 6r$ and

$$V = \pi r^2 h = \frac{1}{36}\pi h^3.$$

- Differentiating this equation with respect to t, we get

$$\frac{dV}{dt} = \frac{1}{12}\pi h^2 \frac{dh}{dt}.$$

- When $dV/dt = 1/2$ and $h = 4$, we have

$$\frac{1}{2} = \frac{4\pi}{3}\frac{dh}{dt},$$

  so the rate at which the height is increasing is

$$\frac{dh}{dt} = \frac{3}{8\pi} \text{ inches/hour.}$$

4. [15%] (a) Use the chain rule and the formula for the derivative of the arcsine to compute the derivative of the function

$$f(x) = \cos\left(\sin^{-1} x\right).$$

(b) Compute the derivative of the function

$$g(x) = \sqrt{1 - x^2}.$$

(c) What is the relationship between your answers in (a) and (b)? Explain why this happens.

Solution.

- (a) Using the chain rule, the derivative of the arcsine, and the definition of the arcsine, we get

$$\begin{aligned} f'(x) &= -\sin(\sin^{-1} x)\frac{d}{dx}\sin^{-1} x \\ &= -\sin(\sin^{-1} x)\frac{1}{\sqrt{1-x^2}} \\ &= -\frac{x}{\sqrt{1-x^2}}. \end{aligned}$$

- (b) Using the chain rule, we get

$$g'(x) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

- (c) The answers in (a) and (b) are the same. This is because

$$\cos(\sin^{-1} x) = \sqrt{1-x^2},$$

  so the original functions in (a) and (b) are the same.

Remark. To show the identity in (c), suppose that $-1 \leq x \leq 1$ and let $\theta = \sin^{-1} x$. Then $\sin\theta = x$ and $-\pi/2 \leq \theta \leq \pi/2$. Using the Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1,$$

we get

$$\cos(\sin^{-1} x) = \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2},$$

where we take the positive square root because $\cos\theta \geq 0$ for $-\pi/2 \leq \theta \leq \pi/2$.

5. [15%] Define a function $f(x)$ by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Use differentiation rules to compute the derivative $f'(x)$ when $x \neq 0$.
(b) Use the definition of the derivative to write down an expression for $f'(0)$ as a limit.
(c) Is $f(x)$ differentiable at $x = 0$? If so, what is $f'(0)$?
Hint. You can assume the following limit: $\lim_{x\to\infty} xe^{-x^2} = 0$.

Solution.

- (a) If $x \neq 0$, then
$$f'(x) = \frac{2}{x^3}e^{-1/x^2}.$$
- (b) The limit definition of the derivative gives
$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{e^{-1/h^2} - 0}{h} \\ &= \lim_{h\to 0} \frac{e^{-1/h^2}}{h}. \end{aligned}$$
- (c) Writing $1/h = t$ and using the limit given in the hint, we find that
$$\lim_{h\to 0^+} \frac{e^{-1/h^2}}{h} = \lim_{t\to\infty} te^{-t^2} = 0.$$
Similarly, writing $1/h = -t$, we find that
$$\lim_{h\to 0^-} \frac{e^{-1/h^2}}{h} = -\lim_{t\to\infty} te^{-t^2} = 0.$$
Since both the left and right limits exists and are equal, it follows that
$$\lim_{h\to 0} \frac{e^{-1/h^2}}{h} = 0,$$
so $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.