CALCULUS
Math 21A, Fall 2015
Sample Midterm 1: Solutions

1. [20%] Say if the following statements are true or false. (For this question only, you don't have to explain your answers.)
(a) If $\lim_{x\to 0} f(x) = 7$, then $f(0) = 7$.
(b) If $\lim_{x\to 0} f(x) = 1$, then $f(x) > 0$ for all nonzero x that are sufficiently close to 0.
(c) If $f(x)$ is a function with domain $[0, 1]$ and $f(0) = -1$, $f(1) = 2$, then $f(x) = 0$ for some x in (0,1).
(d) If $\lim_{x\to 0^+} f(x) = 7$, then $\lim_{x\to 0} f(x^2) = 7$.

Solution.

- Explanations are included in these solutions, although the question doesn't ask for them.
- (a) False. (The existence of the limit as $x \to 0$ does not imply that 0 is in the domain of the function. Moreover, even if 0 is in the domain of the function, the function may be discontinuous at 0, in which case the value of the function at 0 is not equal the limit.)
- (b) True. (The function values $f(x)$ are arbitrarily close to 1 for all nonzero x sufficiently close to 0, so they must be positive. More precisely, taking $\epsilon = 1/2$ — say — in the definition of the limit, we get that there exists $\delta > 0$ such that $|f(x) - 1| < 1/2$ for all x such that $0 < |x| < \delta$, which implies that $f(x) > 1/2 > 0$.)
- (c) False. (For example, the statement is not true for the function

$$f(x) = \begin{cases} -1 & \text{if } 0 \le x \le 1/2, \\ 2 & \text{if } 1/2 < x \le 1. \end{cases}$$

  The statement would be true if $f(x)$ is required to be continuous, by the intermediate value theorem.)
- (d) True. (Writing $t = x^2$, we have $t \to 0^+$ as $x \to 0$ since $x^2 \to 0$ as $x \to 0$ and $x^2 > 0$ for $x \neq 0$. It follows that $\lim_{x\to 0} f(x^2) = \lim_{t\to 0^+} f(t)$.)

2. [30%] Evaluate the following limits or say if they do not exist:

$$\text{(a)}\ \lim_{x\to 2}\frac{x^2-4}{x^2-x-2};$$
$$\text{(b)}\ \lim_{x\to 1}\ln\left[e^x+\ln\left(3-\frac{\tan(2x)}{x}\right)\right];$$
$$\text{(c)}\ \lim_{x\to 0^+}\frac{\sin(\sqrt{x})}{x};$$
$$\text{(d)}\ \lim_{x\to\infty}\frac{1}{x-\sqrt{x^2+x}}.$$

Solution.

- (a) We have

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x^2-x-2} &= \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x+1)}\\ &= \lim_{x\to 2}\frac{x+2}{x+1}\\ &= \frac{4}{3}.\end{aligned}$$

- (b) As the problem is written, we get from the continuity of the logarithm, exponential, and tangent functions that

$$\lim_{x\to 1}\ln\left[e^x+\ln\left(3-\frac{\tan(2x)}{x}\right)\right] = \ln\left[e+\ln\left(3-\tan 2\right)\right].$$

  This result is valid since $3-\tan 2\approx 5.19$ and $e+\ln(3-\tan 2)\approx 4.36$ are both positive and in the domain where ln is continuous.

- As the problem was meant to be written, we have

$$\lim_{x\to 0}\ln\left[e^x+\ln\left(3-\frac{\tan(2x)}{x}\right)\right] = \ln\left[1+\ln\left(3-\lim_{x\to 0}\frac{\tan(2x)}{x}\right)\right].$$

  Using the limit

$$\lim_{x\to 0}\frac{\tan(2x)}{x} = 2\lim_{x\to 0}\frac{\tan(2x)}{2x} = 2,$$

and the fact that $\ln 1 = 0$, we then get that

$$\lim_{x\to 0} \ln\left[e^x + \ln\left(3 - \frac{\tan(2x)}{x}\right)\right] = 0.$$

- (c) Using the limit $\sin\theta/\theta \to 1$ as $\theta \to 0$, we get that

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\sin\sqrt{x}}{x} &= \lim_{x\to 0^+} \left(\frac{1}{\sqrt{x}} \frac{\sin\sqrt{x}}{\sqrt{x}}\right) \\
&= \lim_{x\to 0^+} \left(\frac{1}{\sqrt{x}}\right) \lim_{x\to 0^+} \left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right) \\
&= \lim_{x\to 0^+} \frac{1}{\sqrt{x}} \\
&= \infty,
\end{aligned}$$

so the limit does not exist.

- (d) Using the difference of two squares to remove the square-root in the denominator and simplifying the result, we get

$$\begin{aligned}
\lim_{x\to\infty} \frac{1}{x - \sqrt{x^2+x}} &= \lim_{x\to\infty} \frac{x + \sqrt{x^2+x}}{(x+\sqrt{x^2+x})(x-\sqrt{x^2+x})} \\
&= \lim_{x\to\infty} \frac{x+\sqrt{x^2+x}}{x^2 - (x^2+x)} \\
&= -\lim_{x\to\infty} \frac{x+\sqrt{x^2+x}}{x} \\
&= -\lim_{x\to\infty} \left(1 + \sqrt{1+1/x}\right) \\
&= -2.
\end{aligned}$$

3. [20%] Define a function $f(x)$ with domain all real numbers x by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \sin(\pi/x) & \text{if } 0 < x < 2, \\ x & \text{if } x \geq 2. \end{cases}$$

At what points is $f(x)$ continuous and at what points is $f(x)$ discontinuous? What kinds of discontinuity does $f(x)$ have?

Solution.

- The function $f(x)$ is continuous at $x \neq 0, 2$, since 0, x, and $\sin(\pi/x)$ are continuous functions for $x \neq 0$.

- For $x = 0$, we have

  $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0,$$

  and

  $$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sin(\pi/x)$$

  does not exist, since $\sin(\pi/x)$ oscillates infinitely often between -1 and 1 as $x \to 0^+$. It follows that $\lim_{x \to 0} f(x)$ does not exist, so $f(x)$ is not continuous at 0, where it has an oscillatory discontinuity.

- For $x = 2$, we have

  $$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \sin(\pi/x) = \sin(\pi/2) = 1,$$
  $$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x = 2.$$

  Since the left and right limits are different, $\lim_{x \to 2} f(x)$ does not exist, and $f(x)$ has a jump discontinuity at $x = 2$.

4. [15%] (a) Write an expression for the slope of the tangent line to the graph $y = x^3$ at $x = 1$.
(b) Find the slope of the tangent line in (a).

Solution.

- (a) The slope m of the tangent line is the limit as $x \to 1$ of the slope of the chords between the points $(1, 1)$ and (x, x^3) on the graph:

  $$m = \lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

  Equivalently, writing $x = 1 + h$, we have

  $$m = \lim_{h \to 0} \frac{(1 + h)^3 - 1}{h}$$

- (b) Factoring the numerator and denominator in the first expression and simplifying the result, we get

  $$\begin{aligned} m &= \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \\ &= \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} \\ &= \lim_{x \to 1} x^2 + x + 1 \\ &= 3. \end{aligned}$$

  Alternatively, expanding the cube in the second expression, we get

  $$\begin{aligned} m &= \lim_{h \to 0} \frac{(1 + h)^3 - 1}{h} \\ &= \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h} \\ &= \lim_{h \to 0} 3 + 3h + h^2 \\ &= 3. \end{aligned}$$

5. [15%] (a) Suppose that a function $f(x)$ is defined for all x in an interval about c, except possibly at c itself. Give the precise ϵ-δ definition of

$$\lim_{x \to c} f(x) = L.$$

(b) Use the ϵ-δ definition to prove that

$$\lim_{x \to 0} (3 - 7x^2) = 3.$$

Solution.

- (a) $\lim_{x \to c} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

  $$0 < |x - c| < \delta \text{ implies that } |f(x) - L| < \epsilon.$$

- (b) Let $\epsilon > 0$ be given and choose

  $$\delta = \sqrt{\frac{\epsilon}{7}} > 0.$$

  Then $0 < |x| < \delta$ implies that

  $$\begin{aligned} |(3 - 7x^2) - 3| &= 7|x|^2 \\ &< 7\delta^2 \\ &< \epsilon, \end{aligned}$$

  which proves the result.