Calculus
Math 21A, Fall 2015
Sample Midterm 2: Solutions

1. [20%] Compute the derivatives of the following functions. You do not need to simplify your answer.

(a) $3\sqrt{1+\tan x} - 2\sqrt{\cos x}$ (b) $\dfrac{xe^{2x}}{\ln x}$

(c) $\tan^{-1}(x^2)\sin^2 x.$ (d) $(\tan^{-1} x)^2 \sin(x^2).$

Solution.

- (a)

$$\frac{d}{dx}\left[3\sqrt{1+\tan x} - 2\sqrt{\cos x}\right] = \frac{3}{2\sqrt{1+\tan x}} \cdot \sec^2 x - \frac{2}{2\sqrt{\cos x}}(-\sin x)$$

- (b)

$$\frac{d}{dx}\left[\frac{xe^{2x}}{\ln x}\right] = \frac{(e^{2x} + x \cdot 2e^{2x})\ln x - (1/x) \cdot xe^{2x}}{\ln^2 x}$$

- (c)

$$\frac{d}{dx}\left[\tan^{-1}(x^2)\sin^2 x\right] = \frac{1}{1+x^4} \cdot 2x \cdot \sin^2 x + \tan^{-1}(x^2) \cdot 2\sin x \cdot \cos x$$

- (d)

$$\frac{d}{dx}\left[(\tan^{-1} x)^2 \sin(x^2)\right] = 2\tan^{-1} x \cdot \frac{1}{1+x^2} \cdot \sin(x^2) + (\tan^{-1} x)^2 \cdot \cos(x^2) \cdot 2x$$

2. [15%] (a) State the definition of the derivative $f'(c)$ of a function $f(x)$ at $x = c$.
(b) Suppose that $f(x) = 1/\sqrt{x}$. Compute $f'(9)$ from the definition of the derivative. (No credit for using differentiation rules.)

Solution.

- (a)
$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}.$$

- (b)
$$\begin{aligned} f'(9) &= \lim_{h\to 0} \frac{1/\sqrt{9+h} - 1/\sqrt{9}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{9} - \sqrt{9+h}}{\sqrt{9}\cdot\sqrt{9+h}\cdot h} \\ &= \lim_{h\to 0} \frac{(3-\sqrt{9+h})(3+\sqrt{9+h})}{3\sqrt{9+h}\cdot(3+\sqrt{9+h})\cdot h} \\ &= \lim_{h\to 0} \frac{-h}{3\sqrt{9+h}\cdot(3+\sqrt{9+h})\cdot h} \\ &= -\lim_{h\to 0} \frac{1}{3\sqrt{9+h}\cdot(3+\sqrt{9+h})} \\ &= -\frac{1}{3\cdot 3\cdot(3+3)} \\ &= -\frac{1}{54}. \end{aligned}$$

- This answer agrees with the power rule:
$$\begin{aligned} \frac{d}{dx}x^{-1/2} &= -\frac{1}{2}x^{-3/2} \\ &= -\frac{1}{54} \qquad \text{at } x = 9. \end{aligned}$$

3. [20%] At what point does the tangent line to the curve

$$2x^2 + y^3 = x^3 + y^2$$

at $(x, y) = (2, 1)$ intersect the x-axis?

Solution.

- Differentiating the equation of the curve with respect to x, we get that

$$4x + 3y^2 \frac{dy}{dx} = 3x^2 + 2y \frac{dy}{dx}.$$

- Evaluation of this equation at $(x, y) = (2, 1)$ gives

$$8 + 3\frac{dy}{dx} = 12 + 2\frac{dy}{dx},$$

  so the slope of the tangent line is

$$\frac{dy}{dx} = 4.$$

- The equation of the tangent line is

$$y - 1 = 4(x - 2).$$

- The line intersect the x-axis when $y = 0$, so $x = 7/4$ and the point of intersection is

$$(x, y) = (7/4, 0).$$

4. [15%] Suppose that a particle moves a distance s after time t where

$$s = t - 2t^2 + t^3.$$

(a) At what times is the velocity of the particle equal to zero?
(b) How far has the particle moved when its acceleration is zero?

Solution.

- The velocity of the particle is

$$\frac{ds}{dt} = 1 - 4t + 3t^2 = (1 - 3t)(1 - t).$$

- The velocity is zero when $t = 1/3$ or $t = 1$.
- The acceleration of the particle is

$$\frac{d^2s}{dt^2} = -4 + 6t,$$

which is zero if $t = 2/3$. The location of the particle at that time is $s = 2/27$.

- To answer the question as stated, the particle moves forward from $s = 0$ to $s = 4/27$ for $0 \leq t \leq 1/3$, when its velocity is positive, then it moves backward from $s = 4/27$ to $s = 2/27$ for $1/3 \leq t \leq 2/3$, when its velocity is negative. The total distance moved by the particle is therefore

$$\frac{4}{27} + \frac{2}{27} = \frac{2}{9}.$$

5. [10%] Suppose that a function $f(x)$ has derivative $f'(x) = e^{x^2}$ and

$$g(x) = f(\sqrt{\sin x}).$$

Compute $g'(x)$

Solution.

- By the chain rule,

$$g'(x) = f'(\sqrt{\sin x}) \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x.$$

- Using the expression for $f'(x)$ we then get that

$$g'(x) = e^{\sin x} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x.$$

6. [20%] A conical tank of height 3 m and radius 5 m at its top is filled with water at a rate of $0.2\,\mathrm{m}^3\mathrm{s}^{-1}$. Find the rate at which the height of the water is increasing when the height is 1 m.

Hint. The volume V of a cone with height h and radius r is $V = \frac{1}{3}\pi r^2 h$.

Solution.

- Let $h(t)$ and $r(t)$ be the height of the water and the radius of the surface of the water at time t, respectively.

- By similarity $r/h = 5/3$, so $r = 5h/3$ and

$$V = \frac{25}{27}\pi h^3.$$

- Differentiating this equation with respect to t, we get

$$\frac{dV}{dt} = \frac{25}{9}\pi h^2 \frac{dh}{dt}.$$

- Evaluating this equation at $h = 1$ and $dV/dt = 1/5$ and solving for dh/dt, we get that

$$\frac{dh}{dt} = \frac{9}{125\pi}\,\mathrm{ms}^{-1}.$$