CALCULUS: **Math 21C, Fall 2010**
Final Exam: Solutions

1. [25 pts] Do the following series converge or diverge? State clearly which test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ (e) $\displaystyle\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$

Solution.

- (a) The series diverges by comparison with the divergent harmonic series $\sum 1/n$. Either use the limit comparison test:

$$\lim_{n\to\infty} \frac{1/\sqrt{n(n+1)}}{1/n} = \frac{1}{\sqrt{1+1/n}} = 1,$$

  or the direct comparison test:

$$\frac{1}{\sqrt{n(n+1)}} > \frac{1}{\sqrt{(n+1)^2}} = \frac{1}{n+1}.$$

  Note that the reverse inequality

$$\frac{1}{\sqrt{n(n+1)}} < \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

  doesn't lead to any conclusion.

- (b) The series converges by the alternating series test since $\{1/\sqrt{n}\}$ is a decreasing positive sequence whose limit as $n \to \infty$ is zero.

- (c) The series converges by the ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3/2^{n+1}}{n^3/2^n} = \lim_{n\to\infty} \frac{(1+1/n)^3}{2} = \frac{1}{2} < 1.$$

  Or you can use the root test and the fact that $\lim_{n\to\infty} n^{1/n} = 1$.

- (d) The limit of the terms

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$$

is nonzero, so the series diverges by the nth term test.

- (e) The series is a telescoping series of the form $\sum(b_n - b_{n+1})$ with $b_n = \cos(1/n)$. The limit of the sequence $\{b_n\}$ exists, so the series converges by the telescoping series test. In fact,

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right] = \cos(1) - 1.$$

2. [20 pts] Determine the interval of convergence (including the endpoints) for the following power series. State explicitly for what values of x the series converges absolutely, converges conditionally, or diverges. Specify the radius of convergence R and the center of the interval of convergence a.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{(n+1)^2}(x+2)^n.$$

Solution.

- Applying the ratio test, we find that the series converges absolutely if

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(-1)^{n+1}3^{n+1}(x+2)^{n+1}/(n+2)^2}{(-1)^n 3^n (x+2)^n/(n+1)^2}\right| &= 3|x+2| \lim_{n\to\infty}\left(\frac{1+1/n}{1+2/n}\right)^2 \\ &= 3|x+2| \\ &< 1.\end{aligned}$$

  Thus, the series converges absolutely if

$$|x+2| < \frac{1}{3}$$

  and diverges if $|x+2| > 1/3$. The radius of convergence and center of the interval of convergence are

$$R = \frac{1}{3}, \qquad a = -2.$$

- At the endpoints $x = -5/3$, $x = -7/3$ the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}, \qquad \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

  respectively. These are absolutely convergent p series (with $p = 2 > 1$).

- Summarizing, the power series converges absolutely if

$$|x+2| \le \frac{1}{3}$$

  and diverges otherwise.

3. [20 pts] (a) Write the Taylor polynomial $P_2(x)$ at $x = 0$ of order 2 for the function

$$f(x) = e^{-x}.$$

(b) Use Taylor's theorem with remainder to give a numerical estimate of the maximum error in approximating $e^{-0.1}$ by $P_2(0.1)$. Is $P_2(0.1)$ an overestimate or an underestimate of the actual value of $e^{-0.1}$?

Solution.

- (a) The Taylor series for e^x is

$$e^x = 1 + x + \frac{1}{2}x^2 + \dots$$

so replacing x by $-x$ we find that the Taylor series for e^{-x} is

$$e^x = 1 - x + \frac{1}{2}x^2 + \dots$$

and

$$P_2(x) = 1 - x + \frac{1}{2}x^2.$$

Alternatively, you can note that

$$f'(x) = -e^{-x}, \qquad f''(x) = e^{-x}$$

and use the formula for the Taylor coefficients

$$c_0 = f(0) = 1, \qquad c_1 = f'(0) = -1, \qquad c_2 = \frac{f''(0)}{2!} = \frac{1}{2}.$$

Then

$$P_2(x) = c_0 + c_1 x + c_2 x^2 = 1 - x + \frac{1}{2}x^2.$$

- (b) According to Taylor's theorem with remainder,

$$f(0.1) = P_2(0.1) + \frac{f'''(c)}{3!}(0.1)^3$$

for some $0 < c < 0.1$. We have $f'''(x) = -e^{-x}$, so

$$e^{-0.1} = P_2(0.1) - \frac{1}{6}e^{-c}(0.1)^3.$$

- The error is negative, so $P_2(0.1)$ is an *overestimate* of $e^{-0.1}$.
- Since $c > 0$, we have $0 < e^{-c} < 1$ and therefore

$$0 < P_2(0.1) - e^{-0.1} < \frac{(0.1)^3}{6}$$

and the maximum error is

$$\frac{10^{-3}}{6}.$$

4. [20 pts] Suppose that

$$\vec{u} = 3\vec{i} - \vec{j} + 2\vec{k}, \qquad \vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$$

(a) Find the angle θ between $\vec{u}$ and $\vec{v}$. (You can express the answer as an inverse trigonometric function.)
(b) Find the directional derivatives of the function

$$f(x, y, z) = \frac{x}{y+z}$$

at the point $(4, 1, 1)$ in the directions $\vec{u}$, $\vec{v}$ given in (a).

Solution.

- (a) Using the dot product, we get

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{6-1-4}{\sqrt{9+1+4}\sqrt{4+1+4}} = \frac{1}{3\sqrt{14}},$$

  so

$$\theta = \cos^{-1}\left(\frac{1}{3\sqrt{14}}\right).$$

- (b) The gradient of f is

$$\nabla f(x, y, z) = \frac{1}{y+z}\vec{i} - \frac{x}{(y+z)^2}\vec{j} - \frac{x}{(y+z)^2}\vec{k}$$

  The gradient at $(4, 1, 1)$ is therefore

$$\nabla f(4, 1, 1) = \frac{1}{2}\vec{i} - \vec{j} - \vec{k}.$$

  The directional derivatives of f at $(4, 1, 1)$ in the directions $\vec{u}$, $\vec{v}$ are

$$\nabla f(4,1,1)\cdot\vec{u} = \frac{3}{2} + 1 - 2 = \frac{1}{2},$$
$$\nabla f(4,1,1)\cdot\vec{v} = 1 - 1 + 2 = 2.$$

5. [20 pts] Find a parametric equation for the line through the point $(1, 2, 3)$ whose direction vector is orthogonal to both $\vec{v}$ and $\vec{w}$ where

$$\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}, \qquad \vec{w} = \vec{i} + 3\vec{j} + \vec{k}$$

Solution.

- A direction vector of the line is $\vec{u} = \vec{v} \times \vec{w}$ or

$$\vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 1 & 3 & 1 \end{vmatrix} = 7\vec{i} - 4\vec{j} + 5\vec{k}$$

- The parametric equation of the line is

$$\vec{r} = \vec{r}_0 + t\vec{u}$$

where $\vec{r}_0 = \vec{i} + 2\vec{j} + 3\vec{k}$ is the position vector of $(1, 2, 3)$ or

$$x = 1 + 7t, \qquad y = 2 - 4t, \qquad z = 3 + 5t.$$

6. [20 pts] Find an equation for the tangent plane to the surface

$$x^3 + y \sin z = 1$$

at the point $(1, 1, 0)$.

Solution.

- A normal vector $\vec{n}$ of the level surface $x^3 + y \sin z = 1$ is the gradient of $f(x, y, z) = x^3 + y \sin z$. We have

$$\nabla f(x, y, z) = 3x^2 \vec{i} + \sin z \vec{j} + y \cos z \vec{k}.$$

  Evaluating this vector at the point $(1, 1, 0)$, we get $\vec{n} = 3\vec{i} + \vec{k}$.

- The equation of the plane is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

  where $\vec{r}_0 = \vec{i} + \vec{j}$ is the position vector of $(1, 1, 0)$. This gives

$$3\,(x - 1) + z = 0$$

  or

$$3x + z = 3.$$

7. [25 pts] (a) Find all critical points of the function

$$f(x, y) = x^3 + 3xy + y^3.$$

(b) Classify the critical points as local maximums, local minimums, or saddle-points.

Solution.

- (a) The critical points satisfy $f_x = 0$, $f_y = 0$ or

  $$3x^2 + 3y = 0, \qquad 3x + 3y^2 = 0.$$

  It follows that $y = -x^2$ and $x + x^4 = 0$, so $x = 0$ or $x = -1$. The corresponding y-values are $y = 0$ or $y = -1$. The critical points are therefore

  $$(0, 0), \qquad (-1, -1).$$

- (b) The second-order partial derivatives of f are

  $$f_{xx} = 6x, \qquad f_{yy} = 6y, \qquad f_{xy} = 3,$$

  so

  $$f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9.$$

  At $(0, 0)$, we have $f_{xx}f_{yy} - f_{xy}^2 = -9 < 0$, so this is a saddle point. At $(-1, -1)$, we have $f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} = -6 < 0$ so this is a local maximum.

8. [20 pts] Suppose that

$$D = \{(x, y) : 0 \le x \le 1,\, 0 \le y \le 1\}$$

is the closed unit square and

$$f(x, y) = 3x - 2y + 1.$$

Find the global maximum and minimum values of $f : D \to \mathbb{R}$. At what points (x, y) in D does f attain its maximum and minimum?

Solution.

- The function is differentiable and $\nabla f = 3\vec{\imath} - 2\vec{\jmath}$ is never zero, so f cannot attain its maximum or minimum values inside D and must attain them on the boundary.

- On the side $x = 0$, $0 \le y \le 1$, we have $f(0, y) = -2y + 1$. The derivative of this function with respect to y is nonzero, so it must attain its maximum and minimum on the side at the endpoints $y = 0$, $y = 1$. A similar argument applies to the other three sides where $f(1, y) = -2y + 4$, $f(x, 0) = 3x + 1$, $f(x, 1) = 3x - 1$.

- It follows that f attains its maximum and minimum values at one of the corners of the square. Since

  $$f(0, 0) = 1, \qquad f(1, 0) = 4, \qquad f(0, 1) = -1, \qquad f(1, 1) = 2$$

  we see that the global maximum of f is 4, attained at $(1, 0)$, and the global minimum of f is -1, attained at $(0, 1)$.

9. [20 pts] Find the maximum and minimum values of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraint

$$x^4 + y^4 + z^4 = 1.$$

Solution.

- According to the method of Lagrange multipliers, the maximum and minimum values are attained at points (x, y, z) such that $\nabla f = \lambda \nabla g$ and $g = 0$ where

  $$g(x, y, z) = x^4 + y^4 + z^4 - 1.$$

  It follows that

  $$2x = 4\lambda x^3, \quad 2y = 4\lambda y^3, \quad 2z = 4\lambda z^3, \quad x^4 + y^4 + z^4 = 1.$$

- The possible values for x are $x = 0$ and $x = \pm c$ where

  $$c = \frac{1}{\sqrt{2\lambda}}$$

  and similarly for y and z.

- All three of (x, y, z) are cannot be zero, since then $x^4 + y^4 + z^4 = 0$ and the constraint is not satisfied.

- If two of (x, y, z) are zero and one is equal to $\pm c$, then

  $$x^4 + y^4 + z^4 = c^4$$

  so $c^4 = 1$. It follows that $c = \pm 1$ and

  $$x^2 + y^2 + z^2 = c^2 = 1.$$

- If one of (x, y, z) is zero and the other two are equal to $\pm c$, then

  $$x^4 + y^4 + z^4 = 2c^4$$

  so $c^4 = 1/2$. It follows that $c = \pm 1/2^{1/4}$ and

  $$x^2 + y^2 + z^2 = 2c^2 = \sqrt{2}.$$

- If all three of (x, y, z) are equal to $\pm c$, then

$$x^4 + y^4 + z^4 = 3c^4$$

so $c^4 = 1/3$. It follows that $c = \pm 1/3^{1/4}$ and

$$x^2 + y^2 + z^2 = 3c^2 = \sqrt{3}.$$

- The smallest of these values of f is 1 and the largest is $\sqrt{3}$. It follows that the minimum value of f subject to the constraint $g = 0$ is 1, which is attained at the points

$$(x, y, z) = (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1),$$

and the maximum value is $\sqrt{3}$, which is attained at the points

$$(x, y, z) = \left(\pm\frac{1}{3^{1/4}}, \pm\frac{1}{3^{1/4}}, \pm\frac{1}{3^{1/4}}\right).$$

Here, any combination of signs is allowed, so there are $2^3 = 8$ such points.

10. [10 pts] The Fibonacci sequence $\{a_n\}$

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

is defined by the recursion relation

$$a_{n+2} = a_{n+1} + a_n \qquad \text{for } n \geq 1,$$

meaning that we add the two preceding terms to get the next term, starting with $a_1 = 1$, $a_2 = 1$. Although the Fibonacci sequence diverges, the sequence $\{b_n\}$ of ratios of successive terms

$$b_n = \frac{a_{n+1}}{a_n}$$

or

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots$$

converges. Find

$$b = \lim_{n\to\infty} b_n.$$

Hint. Write down a recursion relation for the b_n's and take the limit as $n \to \infty$. You can assume that the limit b of the sequence $\{b_n\}$ exists.

Solution.

- Dividing the recursion relation for a_n by a_{n+1}, we get

$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}}.$$

  Writing this in terms of b_n, we get that

$$b_{n+1} = 1 + \frac{1}{b_n}.$$

  Taking the limit of this equation as $n \to \infty$, we obtain

$$b = 1 + \frac{1}{b}$$

  It follows that b satisfies the quadratic equation

$$b^2 - b - 1 = 0$$

whose solutions are

$$b = \frac{1 \pm \sqrt{5}}{2}.$$

Since $b_n \geq 0$, we must have $b \geq 0$, so b is the positive solution

$$b = \frac{1 + \sqrt{5}}{2} \approx 1.618.$$

- **Remark.** The limit b is normally written as ϕ and called the 'golden ratio.' It has the property that

  $$\frac{1}{\phi} = \phi - 1$$

  meaning that the reciprocal of ϕ is $\phi - 1$. Geometrically this means that if you remove a square from a rectangle whose sides are in the ratio ϕ, then the remaining rectangle has sides in the same ratio as the original one; because of this property, the ancient Greeks regarded ϕ as giving the most harmonious proportions.

  Fibonacci (c. 1170 – 1250), or Leonardo of Pisa, was a medieval scholar who traveled widely around the Mediterranean. He translated many Arabic mathematical works into Latin and introduced modern Arabic numerals in place of Roman numerals to Europe (where mathematics had essentially died out since classical times).