Calculus: Math 21C, Fall 2010
Midterm 2: Solutions

1. [20%] Determine the interval of convergence (including the endpoints) for the following power series. State for what values of x the series converges absolutely, converges conditionally, or diverges. In each case, specify the radius of convergence R and the center of the interval of convergence a.

$$\text{(a)} \quad \sum_{n=1}^{\infty}(-1)^n n^2 (x+1)^n; \qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}(3x-1)^n.$$

Solution.

- (a) Writing the nth term in the power series as a_n, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2(x+1)^{n+1}}{(-1)^n n^2 (x+1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)^2(x+1)}{n^2}\right| \\ &= |x+1| \lim_{n\to\infty}\frac{(n+1)^2}{n^2} \\ &= |x+1| \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 \\ &= |x+1|.\end{aligned}$$

  By the ratio test, the series converges absolutely if $|x+1| < 1$, or $-2 < x < 0$, and it diverges if $|x+1| > 1$. The center of the interval of convergence is $a = -1$ and the radius of convergence is $R = 1$.

- At the endpoints $x = -2, 0$ the series are

$$\sum_{n=1}^{\infty} n^2, \qquad \sum_{n=1}^{\infty}(-1)^n n^2.$$

  These both diverge by the nth term test (in fact the terms diverge to ∞ as $n \to \infty$).

- (b) We have

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(3x-1)^{n+1}/(n+1)^2 2^{n+1}}{(3x-1)^n/n^2 2^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^2(3x-1)}{2(n+1)^2}\right| \\
&= \frac{|3x-1|}{2}\lim_{n\to\infty}\frac{n^2}{(n+1)^2} \\
&= \frac{|3x-1|}{2}\lim_{n\to\infty}\frac{1}{(1+1/n)^2} \\
&= \frac{|3x-1|}{2}.
\end{aligned}$$

  By the ratio test, the series converges absolutely if $|3x-1| < 2$, or $|x-1/3| < 2/3$, or $-1/3 < x < 1$, and it diverges if $|x-1/3| > 2/3$. The center of he interval of convergence is $a = 1/3$ and the radius of convergence is $R = 2/3$.

- At the endpoints $x = -1/3, 1$ the series are

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

  These are both absolutely convergent p-series ($p = 2$).

2. [20%] (a) Write out the Taylor series at $x = 0$ up to and including the cubic terms (x^3) for the functions: (i) $\sin x$; (ii) $1/(1+x)$. (You can state the answer without proof if you know it.)

(b) Multiply the Taylor series you found in (a) to find the Taylor series at $x = 0$ up to and including cubic terms for the function

$$f(x) = \frac{\sin x}{1+x}.$$

(c) Use your answer from (b) to find $f'''(0)$.

Solution.

- (a) The Taylor series are

$$\sin x = x - \frac{1}{3!}x^3 + \dots,$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots.$$

- (b) Multiplying these series, we get

$$\begin{aligned}\frac{\sin x}{1+x} &= \sin x \cdot \frac{1}{1+x} \\ &= \left(x - \frac{1}{6}x^3 + \dots\right)\left(1 - x + x^2 - x^3 + \dots\right) \\ &= x - x^2 + x^3 - \dots - \frac{1}{6}x^3 + \dots \\ &= x - x^2 + \frac{5}{6}x^3 + \dots.\end{aligned}$$

- (c) By Taylor's theorem, the coefficient of the x^3-term is $f'''(0)/3!$, so

$$\frac{f'''(0)}{6} = \frac{5}{6}$$

and therefore

$$f'''(0) = 5.$$

- **Remark.** This result agrees with what you would get by a (longer) direct calculation:

$$\begin{aligned}
f'(x) &= \frac{\cos x}{1+x} - \frac{\sin x}{(1+x)^2}, \\
f''(x) &= -\frac{\sin x}{1+x} - \frac{2\cos x}{(1+x)^2} + \frac{2\sin x}{(1+x)^3}, \\
f'''(x) &= -\frac{\cos x}{1+x} + \frac{3\sin x}{(1+x)^2} + \frac{6\cos x}{(1+x)^3} - \frac{6\sin x}{(1+x)^4}
\end{aligned}$$

so

$$f'''(0) = -1 + 0 + 6 - 0 = 5.$$

3. [20%] Find an equation for the plane through the points $P(1,1,-1)$, $Q(2,0,2)$ and $R(0,-2,1)$.

Solution.

- Two sides of the triangle are

$$\vec{PQ} = \vec{i} - \vec{j} + 3\vec{k}, \qquad \vec{PR} = -\vec{i} - 3\vec{j} + 2\vec{k}.$$

  The normal vector $\vec{n}$ of the plane is orthogonal to both these vectors, so

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\vec{i} - 5\vec{j} - 4\vec{k}.$$

- The equation of the plane is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

  where $\vec{r}_0 = \vec{OP}$, for example, and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector of an arbitrary point on the plane.

- The coordinate form of the equation is

$$7(x-1) - 5(y-1) - 4(z+1) = 0,$$

  or

$$7x - 5y - 4z = 6.$$

- **Remark.** As a check, note that all three points do lie on this plane:

$$7\cdot 1 - 5\cdot 1 - 4\cdot(-1) = 6, \quad 7\cdot 2 - 5\cdot 0 - 4\cdot 2 = 6, \quad 7\cdot 0 - 5\cdot(-2) - 4\cdot 1 = 6.$$

4. [15%] Let

$$\vec{u} = 2\vec{i} + 10\vec{j} - 11\vec{k}, \qquad \vec{v} = 2\vec{i} + 2\vec{j} + \vec{k}.$$

Find: (a) $|\vec{u}|$ and $|\vec{v}|$; (b) the angle θ between $\vec{u}$ and $\vec{v}$; (c) the projection $\mathrm{proj}_{\vec{v}}\,\vec{u}$ of $\vec{u}$ in the direction $\vec{v}$.

Solution.

- (a) We have

$$|\vec{u}| = \sqrt{2^2 + 10^2 + (-11)^2} = \sqrt{225} = 15,$$
$$|\vec{v}| = \sqrt{2^2 + 2^2 + 1^2} = 3.$$

  (b) We have

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{2\cdot 2 + 10\cdot 2 - 11\cdot 1}{15\cdot 3} = \frac{13}{45}$$

  so

$$\theta = \cos^{-1}\left(\frac{13}{45}\right).$$

- (c) We have

$$\mathrm{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\,\vec{v} = \frac{13}{9}\left(2\vec{i} + 2\vec{j} + \vec{k}\right).$$

5. [15%] Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy}, and f_{yy} of the function

$$f(x, y) = \exp\left(x^2 y^3\right).$$

You do NOT need to simplify your answers.

Solution.

- We have

$$\begin{aligned}
f_x &= 2xy^3 e^{x^2y^3}, \\
f_y &= 3x^2y^2 e^{x^2y^3}, \\
f_{xx} &= 2y^3 e^{x^2y^3} + 2xy^3 \cdot 2xy^3 e^{x^2y^3}, \\
f_{xy} &= 2x \cdot 3y^2 e^{x^2y^3} + 2xy^3 \cdot 3x^2y^2 e^{x^2y^3}, \\
f_{yy} &= 3x^2 \cdot 2y e^{x^2y^3} + 3x^2y^2 \cdot 3x^2y^2 e^{x^2y^3}.
\end{aligned}$$

6. [10%] If $\vec{u}$, $\vec{v}$ are three-dimensional vectors, show that

$$(\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v}) = (\vec{u} \cdot \vec{u})\,(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2 .$$

HINT. Use the geometrical properties of the dot and cross product.

Solution.

- Using $\vec{u} \cdot \vec{u} = |\vec{u}|^2$ and the geometrical formula for the magnitude of a vector product, we get

$$\begin{aligned}(\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v}) &= |\vec{u} \times \vec{v}|^2 \\ &= (|\vec{u}||\vec{v}| \sin\theta)^2 \\ &= |\vec{u}|^2|\vec{v}|^2 \sin^2\theta\end{aligned}$$

  where θ is the angle between $\vec{u}$ and $\vec{v}$.

- Using the geometrical formula for the dot product and the pythagorean identity, we may write the right hand side of the equation as

$$\begin{aligned}(\vec{u} \cdot \vec{u})\,(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2 &= |\vec{u}|^2|\vec{v}|^2 - (|\vec{u}||\vec{v}| \cos\theta)^2 \\ &= |\vec{u}|^2|\vec{v}|^2 \left(1 - \cos^2\theta\right) \\ &= |\vec{u}|^2|\vec{v}|^2 \sin^2\theta.\end{aligned}$$

  So the two sides are equal.

- **Remark.** Written out in components, this identity says that

$$\begin{aligned}&(u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\ &\quad = \left(u_1^2 + u_2^2 + u_3^2\right)\left(v_1^2 + v_2^2 + v_3^2\right) - (u_1v_1 + u_2v_2 + u_3v_3)^2\end{aligned}$$

  which you can verify by multiplying out the terms and simplifying the result. (The above geometrical argument is faster!) This result is called Lagrange's identity (in three dimensions). The two dimensional version

$$(u_1v_2 - u_2v_1)^2 = \left(u_1^2 + u_2^2\right)\left(v_1^2 + v_2^2\right) - (u_1v_1 + u_2v_2)^2$$

  is in the works of the Greek mathematician Diophantus, who lived in Alexandria around 200–300 A.D.