CALCULUS: **Math 21C, Fall 2010**
Summary of basic results for Midterm 2.

1. POWER SERIES

Convergence of power series. A power series centered at $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n. \tag{1}$$

Every power series has a radius of convergence $0 \le R \le \infty$ such that the series converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$. The series may or may not converge at the endpoints of the interval of convergence where $|x-a| = R$.

Ratio test. The power series (1) converges absolutely for all x such that

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| < 1$$

if the limit exists. The radius of convergence R is given by

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

if the limit exists (with the natural conventions for $R = 0$ and $R = \infty$).

Differentiation and integration of power series. If a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is the sum of a power series with nonzero radius of convergence $R > 0$, then $f(x)$ has derivatives of all orders inside the interval of convergence $|x-a| < R$. Its derivative f' is given by differentiating the power series of f term-by-term,

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1},$$

and this power series has the same radius of convergence as the power series for f. Power series for higher-order derivatives of f are obtained by the repeated application of this result. Similarly, the integral of f is given by term-by-term integration of the power series of f

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C,$$

and this power series has the same radius of convergence as the power series for f.

Multiplication of power series. If $f(x)$, $g(x)$ have power series expansions

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n (x-a)^n,$$

which both converge in $|x-a| < R$, then $h(x) = f(x)g(x)$ has the power series expansion

$$h(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad c_n = \sum_{k=0}^{n} a_k b_{n-k},$$

which converges in $|x-a| < R$. Writing out the first few terms explicitly, we have

$$\begin{aligned}
&\left[a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots\right] \\
&\qquad\qquad \cdot \left[b_0 + b_1(x-a) + b_2(x-a)^2 + b_3(x-a)^3 + \dots\right] \\
&= a_0b_0 + (a_0b_1 + a_1b_0)(x-a) + (a_0b_2 + a_1b_1 + a_2b_0)(x-a)^2 \\
&\qquad\qquad\quad + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)(x-a)^3 + \dots.
\end{aligned}$$

Taylor series. If a function $f(x)$ has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

with nonzero radius of convergence, then

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

If a function f is defined in an open interval containing a and has derivatives of all orders at a, then $f^{(n)}(a)/n!$ is called the nth Taylor coefficient of f at a and the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the Taylor series of f at a. (A Taylor series at 0 is also called a Maclaurin series.)

Taylor's theorem with remainder. If a function $f(x)$ has $(n+1)$-derivatives in an open interval containing a, then for any x in that interval

$$f(x) = P_n(x) + R_n(x)$$

where

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

is the Taylor polynomial of f of order n and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x. The Taylor series of f converges to f at x if

$$\lim_{n\to\infty} R_n(x) = 0.$$

Estimate of remainder. If

$$\left|f^{(n+1)}(c)\right| \leq M$$

for all $a < c < x$ (or all $x < c < a$) then

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}.$$

2. Examples of Taylor series

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots \qquad (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots + \frac{(-1)^n}{(2n)!}x^{2n} + \dots \qquad (-\infty < x < \infty)$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \dots \qquad (-\infty < x < \infty)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots \qquad (-1 < x < 1)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots + \frac{(-1)^{n+1}}{n}x^n + \dots \qquad (-1 < x < 1)$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!}x^n + \dots \qquad (-1 < x < 1).$$

3. Vectors

Definition. An n-dimensional vector $\vec{u} = \langle u_1, u_2, \dots, u_n \rangle$ is an n-tuple of real numbers $\{u_1, u_2, \dots, u_n\}$. The zero vector is the vector $\vec{0} = \langle 0, 0, \dots, 0 \rangle$. The addition of vectors

$$\vec{u} = \langle u_1, u_2, \dots, u_n \rangle, \qquad \vec{v} = \langle v_1, v_2, \dots, v_n \rangle$$

is defined by adding their components

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, \dots, u_n + v_n \rangle.$$

The addition of vectors corresponds geometrically to the parallelogram law. The multiplication of a vector $\vec{u}$ by a scalar (*i.e.* a real number) k is defined by

$$k\vec{u} = \langle ku_1, ku_2, \dots, ku_n \rangle.$$

Vectors $\vec{u}$, $\vec{v}$ are parallel if $\vec{u} = k\vec{v}$ for some scalar k or one of them is zero. The length, or magnitude, of a vector $\vec{u}$ is

$$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}.$$

The direction of a nonzero vector $\vec{u}$ is the unit vector $\vec{u}/|\vec{u}|$.

Standard basis vectors. For two-dimensional vectors, we introduce the standard basis vectors

$$\vec{i} = \langle 1, 0 \rangle, \qquad \vec{j} = \langle 0, 1 \rangle,$$

and then

$$\langle u_1, u_2 \rangle = u_1\vec{i} + u_2\vec{j}.$$

For three-dimensional vectors, we introduce the standard basis vectors

$$\vec{i} = \langle 1, 0, 0 \rangle, \qquad \vec{j} = \langle 0, 1, 0 \rangle, \qquad \vec{k} = \langle 0, 0, 1 \rangle,$$

and then

$$\langle u_1, u_2, u_3 \rangle = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}.$$

Below, we consider three-dimensional vectors for definiteness.

Position vectors. The position vector from a point $P(x_1, y_1, z_1)$ to a point $Q(x_2, y_2, z_2)$ is

$$\vec{PQ} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}.$$

The position vector of a point $P(x, y, z)$ relative to the origin $O(0, 0, 0)$ is

$$\vec{OP} = x\vec{i} + y\vec{j} + z\vec{k}.$$

We have $\vec{PQ} + \vec{QR} = \vec{PR}$ and $\vec{QP} = -\vec{PQ}$.

Dot product. The dot product of two vectors

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}, \qquad \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

is the scalar

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Geometrical interpretation of the dot product. The dot product gives lengths and angles. The length $|\vec{u}|$ of a vector $\vec{u}$ is given by

$$|\vec{u}|^2 = \vec{u} \cdot \vec{u} = (u_1)^2 + (u_2)^2 + (u_3)^2.$$

If $0 \le \theta \le \pi$ is the angle between $\vec{u}$ and $\vec{v}$ then

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta.$$

Two vectors $\vec{u}$, $\vec{v}$ are orthogonal (perpendicular) if and only if $\vec{u} \cdot \vec{v} = 0$.

Projections. The (orthogonal) projection of a vector $\vec{u}$ in the direction of a nonzero vector $\vec{v}$ is the vector

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}.$$

We can write

$$\vec{u} = \vec{u}_{\parallel} + \vec{u}_{\perp}, \qquad \vec{u}_{\parallel} = \operatorname{proj}_{\vec{v}} \vec{u}, \quad \vec{u}_{\perp} = \vec{u} - \operatorname{proj}_{\vec{v}} \vec{u}$$

where $\vec{u}_{\parallel}$ is parallel to $\vec{v}$ and $\vec{u}_{\perp}$ is orthogonal to $\vec{v}$.

Cross product. The cross product of two three-dimensional vectors

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}, \qquad \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

is the vector

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\,\vec{i} + (u_3v_1 - u_1v_3)\,\vec{j} + (u_1v_2 - u_2v_1)\,\vec{k}.$$

Note that $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$, so the order of the vectors is important.

Geometrical interpretation of the cross product. The cross product gives areas and normal vectors. The cross product $\vec{u} \times \vec{v}$ of vectors $\vec{u}$, $\vec{v}$ is the vector such that:

1. $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$;

2. $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$ where $0 \le \theta \le \pi$ is the angle between $\vec{u}$ and $\vec{v}$;

3. $\{\vec{u}, \vec{v}, \vec{u} \times \vec{v}\}$ in that order is a right-handed system of vectors.

Note that

$$|\vec{u} \times \vec{v}| = \text{area of parallelogram spanned by } \vec{u},\ \vec{v}.$$

Scalar triple product. The scalar triple product (or box product) of three, three-dimensional vectors

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}, \quad \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}, \quad \vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$$

is the scalar

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

It has the property that

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v}).$$

Geometrical interpretation of the scalar triple product. The scalar triple product gives volumes. If $\{\vec{u}, \vec{v}, \vec{w}\}$ are three-dimensional vectors, then

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \text{oriented volume of the parallelepiped spanned by } \{\vec{u}, \vec{v}, \vec{w}\},$$

meaning that it is equal to the positive volume if $\{\vec{u}, \vec{v}, \vec{w}\}$ is right-handed and minus the volume if $\{\vec{u}, \vec{v}, \vec{w}\}$ is left-handed. Three vectors lie in the same plane if and only if their scalar triple product is zero.

Parametric equation of a line. The line through a point $P_0(x_0, y_0, z_0)$ with position vector $\vec{r}_0 = \vec{OP}_0$ in the direction of a nonzero vector $\vec{u}$ is given parametrically by

$$\vec{r}(t) = \vec{r}_0 + t\vec{u}$$

where $\vec{r}(t) = \vec{OP}(t)$ is the position vector of a point $P(t)$ on the line with parameter value $-\infty < t < \infty$. If $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$, and $P(t)$ has coordinates (x, y, z), then the coordinate form of the equation is

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct.$$

Equation of a plane. The equation for a plane with nonzero normal vector $\vec{n}$ through a point $P_0(x_0, y_0, z_0)$ with position vector $\vec{r}_0 = \vec{OP}_0$ is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0.$$

If $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$, then the Cartesian equation of the plane is

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0$$

or $ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$.

4. Partial derivatives

Definition. If $f(x, y)$ is a function of two variables, then the partial derivatives f_x, f_y of f with respect to x, y respectively are defined by

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}, \quad f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

provided that the limits exist. To compute f_x, differentiate $f(x, y)$ with respect to x treating y as a constant; to compute f_y, differentiate $f(x, y)$ with respect to y treating x as a constant. Partial derivatives of functions of three or more variables are defined analogously.

Higher order partial derivatives. If a function $f(x, y)$ has first-order partial derivatives, then we define its second-order partial derivatives by differentiating twice:

$$\begin{aligned}
f_{xx}(x, y) &= \lim_{h\to 0} \frac{f_x(x + h, y) - f_x(x, y)}{h}, \\
f_{xy}(x, y) &= \lim_{h\to 0} \frac{f_x(x, y + h) - f_x(x, y)}{h}, \\
f_{yx}(x, y) &= \lim_{h\to 0} \frac{f_y(x + h, y) - f_y(x, y)}{h}, \\
f_{yy}(x, y) &= \lim_{h\to 0} \frac{f_y(x, y + h) - f_y(x, y)}{h}
\end{aligned}$$

provided these limits exist. Third and higher order partial derivatives are defined analogously.

Equality of mixed partial derivatives. If the partial derivatives f_x, f_y, f_{xy}, f_{yx} of a function $f(x, y)$ exist and are continuous throughout an open region, then $f_{xy} = f_{yx}$.