Calculus: **Math 21C, Fall 2010**
Summary of topics covered after Midterm 2

1. Functions of several variables

Plots. Graph $z = f(x, y)$ of a function $f(x, y)$ of two variables. Contour plots of level curves $f(x, y) = c$. Level surfaces $f(x, y, z) = c$ of a function of three variables.

Sets, limits and continuity. Open and closed sets in $\mathbb{R}^2$ and $\mathbb{R}^3$. Interior points and boundary points. Limits and continuity of functions of several variables. (Intuitive ideas and examples only. No ϵ-δ definitions or proofs will be required for the final.)

Derivatives. Interpretation of the derivative as the local linear approximation of a function. **Theorem.** If the partial derivatives of a function exist and are continuous, then the function is differentiable.

Chain rule. Chain rule for functions of several variables. For example,

$$\begin{aligned}\frac{d}{dt} f\left(x(t), y(t), z(t)\right) &= \frac{\partial f}{\partial x}\left(x(t), y(t), z(t)\right) \frac{dx}{dt}(t) \\ &+ \frac{\partial f}{\partial y}\left(x(t), y(t), z(t)\right) \frac{dy}{dt}(t) + \frac{\partial f}{\partial z}\left(x(t), y(t), z(t)\right) \frac{dz}{dt}(t).\end{aligned}$$

2. Gradients and the directional derivative

Gradient. If $f(x, y)$ is a differentiable function of two variables, then

$$\nabla f(x, y) = f_x(x, y)\vec{i} + f_y(x, y)\vec{j}.$$

If $f(x, y, z)$ is a function of three variables, then

$$\nabla f(x, y, z) = f_x(x, y, z)\vec{i} + f_y(x, y, z)\vec{j} + f_z(x, y, z)\vec{k}.$$

Directional derivative. The directional derivative of a function at a point P, with position vector $\vec{r} = \vec{OP}$, in the direction $\vec{u}$ is the rate of change of f along the parametric line through $\vec{r}$ with direction vector $\vec{u}$:

$$\left(\frac{df}{dt}\right)_{\vec{u}} (\vec{r}) = \frac{d}{dt} f\left(\vec{r} + t\vec{u}\right)\bigg|_{t=0}.$$

For example,

$$\left(\frac{df}{dt}\right)_{\vec{i}} = f_x, \qquad \left(\frac{df}{dt}\right)_{\vec{j}} = f_y.$$

By the chain rule,

$$\left(\frac{df}{dt}\right)_{\vec{u}} (\vec{r}) = \nabla f\,(\vec{r}) \cdot \vec{u}.$$

Use this formula to compute directional derivatives.

Geometrical interpretation of the gradient. The gradient $\nabla f(P)$ of a function f at a point P is in the direction of most rapid increase of f at P. If $\nabla f(P) \neq 0$, the unit vector in this direction is

$$\vec{u} = \frac{\nabla f(P)}{|\nabla f(P)|}.$$

The magnitude $|\nabla f(P)|$ is the largest value of the directional derivative of f in any unit direction.

Normal vectors. The gradient $\nabla f(P)$ at a point P on a level curve $f(x, y) = c$ in two dimensions, or a level surface $f(x, y, z) = c$ in three dimensions, is a vector that is orthogonal to the level curve, or level surface. For example, use this to find the equation of the tangent plane to a surface $f(x, y, z) = c$ at some point on the surface. If $\nabla f(P) \cdot \vec{u} = 0$, then $\vec{u}$ is tangent to the level curve, or level surface, of f at P.

3. Max/Min problems

Critical points. A critical point of a differentiable function f is a point where $\nabla f = 0$. If a differentiable function f attains a local maximum or minimum at an interior point of its domain, then that point is a critical point of f.

Second derivative test. Suppose that $f(x, y)$ has continuous first and second derivatives and (x_0, y_0) is an interior critical point of f.

1. If $f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ (or equivalently $f_{yy} > 0$) at (x_0, y_0), then f attains a local minimum at (x_0, y_0).

2. If $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$ (or equivalently $f_{yy} < 0$) at (x_0, y_0), then f attains a local maximum at (x_0, y_0).

3. If $f_{xx}f_{yy} - f_{xy}^2 < 0$ then f has a saddle point at (x_0, y_0), which is neither a local maximum nor a local minimum.

If $f_{xx}f_{yy} - f_{xy}^2 = 0$ then there is no conclusion from the test.

Global max/min problems. If $f : D \to \mathbb{R}$ is a continuous function defined on a closed, bounded set D, then f is bounded and attains its global maximum and minimum at some point(s) in D. The maximum and minimum values can only be attained at one of the following types of points:

1. points on the boundary of D;

2. interior points where f is not differentiable;

3. interior critical points where f is differentiable and $\nabla f = 0$.

Lagrange multipliers. To find possible points where a function $f(x, y, z)$ attains a local maximum or minimum subject to the constraint $g(x, y, z) = 0$, look for solutions (x, y, z, λ) of the equations

$$\nabla f = \lambda \nabla g, \qquad g(x, y, z) = 0.$$