Calculus

Math 21C, Fall 2010

Summary of basic results about sequences and series.

1. Sequences

Definition of limit. A sequence $\{a_n\}$ converges to a limit L as $n \to \infty$, written

$$\lim_{n\to\infty} a_n = L, \qquad \text{or } a_n \to L \text{ as } n \to \infty$$

if for every $\epsilon > 0$ there exists a number N such that

$$|a_n - L| < \epsilon \qquad \text{for all } n > N.$$

A sequence that does not converge is said to diverge.

Diverges to infinity. A sequence $\{a_n\}$ diverges to ∞, written

$$\lim_{n\to\infty} a_n = \infty,$$

if for every number M there exists N such that

$$a_n > M \qquad \text{for all } n > N$$

Note that such a sequence does not have a finite limit L, so it is divergent not convergent.

Upper bound criterion. If $\{a_n\}$ is an increasing sequence ($a_{n+1} \geq a_n$) that is bounded from above (there exists a number M such that $a_n \leq M$ for every n) then $\{a_n\}$ converges.

Sandwich theorem. If $a_n \leq b_n \leq c_n$ and the limits

$$\lim_{n\to\infty} a_n = L, \qquad \lim_{n\to\infty} c_n = L$$

exist and are equal, then

$$\lim_{n\to\infty} b_n = L.$$

Continuous functions of sequences. If $\lim_{n\to\infty} a_n = L$ and f is a function that is defined on some open interval containing L and is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

2. Series

Definition of sum. A series

$$\sum_{n=1}^{\infty} a_n$$

converges to a sum s if the sequence $\{s_n\}$ of partial sums

$$s_n = \sum_{k=1}^{n} a_k$$

converges to s as $n \to \infty$, meaning that

$$s = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

n-th term test. If $\sum a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

It follows that if $\lim_{n\to\infty} a_n$ does not exist or is nonzero, then $\sum a_n$ diverges.

Upper bound criterion. A series $\sum a_n$ with positive terms ($a_n \geq 0$) converges if and only if its partial sums are bounded from above.

3. Examples of series

Geometric series. The geometric series with ratio r,

$$\sum_{n=0}^{\infty} r^n,$$

converges if and only $|r| < 1$, meaning that $-1 < r < 1$, and then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

p-series. The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Telescoping series. If

$$a_n = b_{n+1} - b_n$$

then the series $\sum a_n$ converges if and only if the limit

$$B = \lim_{n\to\infty} b_n$$

exists, and then

$$\sum_{n=1}^{\infty} (b_{n+1} - b_n) = B - b_1.$$

4. Tests for convergence of a positive series

Integral test. If $a_n = f(n)$ for $n > N$ where $f : [N, \infty) \to \mathbb{R}$ is a continuous, positive, decreasing function and N is some number, then the series $\sum a_n$ converges if and only if the improper Riemann integral

$$\int_N^\infty f(x)\,dx = \lim_{R\to\infty} \int_N^R f(x)\,dx$$

converges.

Comparison test. Let $\sum a_n$, $\sum b_n$ be positive series.

1. If $\sum b_n$ converges and
$$0 \le a_n \le b_n \qquad \text{for all } n > N$$
for some number N, then $\sum a_n$ converges.

2. If $\sum b_n$ diverges and
$$0 \le b_n \le a_n \qquad \text{for all } n > N$$
for some number N, then $\sum a_n$ diverges.

Limit comparison test. Let $\sum a_n$, $\sum b_n$ be positive series with $b_n > 0$ for all $n > N$, where N is some number.

1. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$
where $0 < c < \infty$, then $\sum a_n$ converges if $\sum b_n$ converges and $\sum a_n$ diverges if $\sum b_n$ diverges.

2. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$
then $\sum a_n$ converges if $\sum b_n$ converges.

3. If
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$
then $\sum a_n$ diverges if $\sum b_n$ diverges.

Ratio test. Let $\sum a_n$ be a positive series such that the limit

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists.

1. If $0 \leq \rho < 1$, then $\sum a_n$ converges.
2. If $\rho > 1$, then $\sum a_n$ diverges.
3. If $\rho = 1$, then there is no conclusion.

Root test. Let $\sum a_n$ be a positive series such that the limit

$$\rho = \lim_{n\to\infty} (a_n)^{1/n}$$

exists.

1. If $0 \leq \rho < 1$, then $\sum a_n$ converges.
2. If $\rho > 1$, then $\sum a_n$ diverges.
3. If $\rho = 1$, then there is no conclusion.

5. Absolutely convergent series

Definition. A series

$$\sum a_n,$$

whose terms a_n may be positive positive or negative, converges absolutely (or is absolutely convergent) if

$$\sum |a_n|$$

converges. A series such that $\sum a_n$ converges but $\sum |a_n|$ diverges is said to converge conditionally (or be conditionally convergent).

Convergence. Every absolutely convergent series is convergent. This means that if one of the tests for positive series shows that $\sum |a_n|$ converges, then $\sum a_n$ converges.

Rearrangements. A series $\sum a_n$ is absolutely convergent if and only if every rearrangement $\sum b_n$ of the series converges to the same sum.

6. Alternating series

Definition. An alternating series is a series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \dots$$

where $\{u_n\}$ is a sequence of positive terms ($u_n \geq 0$).

Alternating series test. If u_n is a decreasing sequence of positive terms such that

$$\lim_{n\to\infty} u_n = 0,$$

then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n$$

converges. (Note that it only converges absolutely if $\sum u_n$ converges, which may or may not be the case.)