Calculus

Math 21C, Fall 2010

Midterm I: Solutions

1. [20%] Do the following sequences $\{a_n\}$ converge or diverge as $n \to \infty$? If a sequence converges, find its limit. Justify your answers.

$$\text{(a)} \quad a_n = 2 + (-1)^n; \qquad \text{(b)} \quad a_n = \frac{n}{e^n}; \qquad \text{(c)} \quad a_n = \left(1 + \frac{2}{n}\right)^n.$$

Solution.

- (a) The sequence diverges since it oscillates between 1 and 3. For example, if $\epsilon = 1$, there is no number L such that $|a_n - L| < \epsilon$ for all sufficiently large n, since then we would have both $|1 - L| < 1$ (or $0 < L < 2$) and $|3 - L| < 1$ or $(2 < L < 4)$, which is impossible. So there is no L that satisfies the definition of the limit.

- (b) Both the numerator n and the denominator e^n are differentiable functions that diverge to ∞ as $n \to \infty$. Differentiating the numerator and denominator with respect to n, we get by l'Hôspital's rule that

$$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0.$$

  Thus, the sequence converges to 0.

- (c) The sequence converges to e^2. To show this, let

$$b_n = \ln a_n = n \ln\left(1 + \frac{2}{n}\right).$$

  Then, writing $n = 1/x$, we have

$$\lim_{n\to\infty} b_n = \lim_{x\to 0} \frac{\ln(1+2x)}{x}.$$

  The fraction is an indeterminate limit $(0/0)$ of differentiable functions, so by l'Hôspital's rule

$$\lim_{x\to 0} \frac{\ln(1+2x)}{x} = \lim_{x\to 0} \frac{2/(1+2x)}{1} = \lim_{x\to 0} \frac{2}{1+2x} = 2.$$

Thus, $b_n \to 2$ as $n \to \infty$. Since $a_n = e^{b_n}$ and e^x is a continuous function, it follows that

$$\lim_{n\to\infty} a_n = e^{\lim_{n\to\infty} b_n} = e^2.$$

- More generally, for any $-\infty < x < \infty$, we have the limit

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

2. [20%] Determine whether the following series converge or diverge. You can use any appropriate test provided that you explain your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$; (c) $\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$;

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$.

Solution.

- (a) The series converges. Let

$$a_n = \frac{n^3}{3^n}.$$

Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)^3/3^{n+1}}{n^3/3^n} \\
&= \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} \\
&= \lim_{n\to\infty} \frac{(1+1/n)^3}{3} \\
&= \frac{1}{3}.
\end{aligned}$$

Since the series is positive and this limit exists and is strictly less than 1, the series converges by the ratio test.

- (b) The series diverges. Let

$$a_n = \frac{n}{n^2+1}, \qquad b_n = \frac{1}{n}.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = 1.$$

Since this limit is finite and nonzero, the positive series $\sum n/(n^2+1)$ and $\sum 1/n$ converge or diverge together by the limit comparison test. Since the harmonic series $\sum 1/n$ diverges (it is a p-series with $p=1$), the series $\sum n/(n^2+1)$ also diverges.

- Alternatively, we can use the direct comparison test. For all $n \geq 1$

$$\frac{n}{n^2+1} \geq \frac{n}{n^2+n^2} = \frac{1}{2n},$$

and the series $\sum 1/(2n)$ diverges, so $\sum n/(n^2+1)$ diverges. Note that the direct comparison test does not give any conclusion from the inequality

$$0 \leq \frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n},$$

because a series bounded from above by a divergent series might converge or diverge.

- (c) The series diverges. Writing $n = 1/x$ and using the standard limit for $\sin x/x$, which can also be evaluated by use of l'Hôspital's rule, we have

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

Since the terms $n\sin(1/n)$ of the series do not converge to zero as $n \to \infty$, the series diverges by the nth term test.

- (d) The series is a telescoping series with terms $a_n = b_n - b_{n+1}$ and $b_n = 1/\sqrt{n}$. Since $\lim_{n\to\infty} b_n$ exists (it's zero) the series converges. Although the question doesn't ask for the value of the sum, it is

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) &= \frac{1}{\sqrt{1}} - \lim_{n\to\infty}\frac{1}{\sqrt{n}} \\ &= 1.\end{aligned}$$

3. [20%] Determine whether the following series converge or diverge and justify your answer.

$$(a)\quad \sum_{n=2}^{\infty} \frac{1}{n \ln n}; \qquad\qquad (b)\quad \sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}.$$

Solution.

- (a) The series diverges. We have

$$\frac{1}{n \ln n} = f(n), \qquad f(x) = \frac{1}{x \ln x}.$$

  The function $f(x)$ is a continuous, positive function on $2 \le x < \infty$. It is decreasing since the denominator $x \ln x$ is increasing. Therefore, by the integral test, the series converges or diverges with the integral

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx.$$

- To evaluate this integral, we make the substitution $u = \ln x$, with $du = dx/x$, which gives

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u}\, du = \left[\ln u\right]_{\ln 2}^{\infty} = \infty$$

  since $\ln u \to \infty$ as $u \to \infty$, so the integral and the series diverge.

- (b) The series converges. It is an alternating series, and the convergence follows from the alternating series test for $\sum(-1)^n u_n$ with

$$u_n = \frac{\ln n}{n}$$

  once we check that $\{u_n\}$ is a positive decreasing sequence that converges to 0 as $n \to \infty$.

- The sequence $\{u_n\}$ is obviously positive for $n \ge 2$.

- It is not immediately obvious that the sequence $\{u_n\}$ is decreasing, since both the numerator $\ln n$ and the denominator n are increasing functions of n. We therefore consider the function

  $$f(x) = \frac{\ln x}{x}.$$

  By the quotient rule for derivatives, we have

  $$f'(x) = \frac{x \cdot (1/x) - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2}.$$

  Since $\ln x > 1$ for $x > e$, we have $f'(x) < 0$ for $e < x < \infty$, so the function $f(x)$ is decreasing there, and the sequence $\{u_n\}$ is decreasing at least for $n \geq 3$.

- To show that $u_n \to 0$ as $n \to \infty$, we note that the numerator $\ln n$ and denominator n are both differenetiable functions of n that approach ∞ as $n \to \infty$, so l'Hôspital's rule applies. Differentiating with respect to n, we get

  $$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

4. [20%] (a) Does the following series diverge, converge conditionally, or converge absolutely? Justify your answer.

$$A = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \cdots. \tag{1}$$

(b) Let

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots.$$

Given that $S = \pi^2/6$, find the sum A in (1).

HINT. Consider $S - A$ and express it in terms of S.

Solution.

- (a) The series

  $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

  converges absolutely, since the series of absolute values

  $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

  is a convergent p-series (with $p = 2 > 1$).

- (b) By the linearity of limits and sums, if the series $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n - b_n)$ converges and

  $$\sum(a_n - b_n) = \sum a_n - \sum b_n.$$

  Also for any constant c, we have

  $$\sum c a_n = c \sum a_n.$$

- Since the series for S and A both converge, we get using the previous result that

$$\begin{aligned}
S - A &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots. - \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots\right) \\
&= (1-1) + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \left(\frac{1}{3^2} - \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{4^2}\right) + \left(\frac{1}{5^2} - \frac{1}{5^2}\right) + \dots \\
&= 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \dots\right) \\
&= 2\left(\frac{1}{2^2} \cdot 1 + \frac{1}{2^2} \cdot \frac{1}{2^2} + \frac{1}{2^2} \cdot \frac{1}{3^2} + \frac{1}{2^2} \cdot \frac{1}{4^2} + \dots\right) \\
&= \frac{2}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right) \\
&= \frac{1}{2} S.
\end{aligned}$$

Solving this equation for A in terms of S, we get $A = S/2$. Since $S = \pi^2/6$, it follows that $A = \pi^2/12$, or

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \dots.$$

- **Remark.** The same trick works for any alternating p-series with $p > 1$ and gives
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = \left(1 - \frac{1}{2^{p-1}}\right) \sum_{n=1}^{\infty} \frac{1}{n^p}.$$
So we can find the sum of an alternating p-series if we know the sum of the corresponding p-series.

5. [20%] (a) State the definition for a sequence $\{a_n\}$ to converge to a limit L as $n \to \infty$.
(b) If

$$a_n = \frac{1}{\sqrt{n}} \qquad \text{for } n = 1, 2, 3, \ldots$$

prove *from the definition* that

$$\lim_{n\to\infty} a_n = 0.$$

(No credit unless you use the definition.)

Solution.

- (a) A sequence $\{a_n\}$ converges to a limit L if for every $\epsilon > 0$ there exists a number N such that

$$|a_n - L| < \epsilon \qquad \text{for every } n > N.$$

- (b) Given $\epsilon > 0$, let $N = 1/\epsilon^2$. Then if $n > N$, we have

$$\begin{aligned} \left| \frac{1}{\sqrt{n}} - 0 \right| &= \frac{1}{\sqrt{n}} \\ &< \frac{1}{\sqrt{N}} \\ &< \epsilon, \end{aligned}$$

since $1/\sqrt{N} = \epsilon$. It follows that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$