Calculus: Math 21C, Fall 2010
Sample Final Questions

1. Do the following sequences $\{a_n\}$ converge or diverge as $n \to \infty$? If a sequence converges, find its limit. Justify your answers.

$$\text{(a)}\quad a_n = \frac{2n^2+3n^3}{2n^3+3n^2}; \qquad \text{(b)}\quad a_n = \cos(n\pi); \qquad \text{(c)}\quad a_n = \frac{\sin(n^2)}{n^2}.$$

2. Do the following series converge or diverge? State clearly which test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+4}{6n-17}$

(b) $\displaystyle\sum_{n=2}^{\infty} \sqrt{\frac{n}{n^4+7}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^{n+1}}{(2n)!}$

(d) $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$

(e) $\displaystyle\frac{1}{1^4} + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} - \frac{1}{9^4} + \cdots$

(f) $\displaystyle\sum_{n=1}^{\infty} \left[e^n - e^{n+1}\right]$

3. Determine the interval of convergence (including the endpoints) for the following power series. State explicitly for what values of x the series converges absolutely, converges conditionally, and diverges. Specify the radius of convergence R and the center of the interval of convergence a.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n} (x-1)^n.$$

4. Write the Taylor polynomial $P_2(x)$ at $x = 0$ of order 2 for the function

$$f(x) = \ln(1+x).$$

Use Taylor's theorem with remainder to give a numerical estimate of the maximum error in approximating $\ln(1.1)$ by $P_2(0.1)$.

5. (a) Find the value(s) of c for which the vectors

$$\vec{u} = c\vec{i} + \vec{j} + c\vec{k}, \qquad \vec{v} = 2\vec{i} - 3\vec{j} + c\vec{k}.$$

are orthogonal.

(b) Find the value(s) of c for which the vectors

$$\vec{u} = c\vec{i} + \vec{j} + c\vec{k}, \qquad \vec{v} = 2\vec{i} - 3\vec{j} + c\vec{k}, \qquad \vec{w} = \vec{i} + 6\vec{k}.$$

lie in the same plane.

6. Find a parametric equation for the line in which the planes

$$3x - 6y - 4z = 15 \text{ and } 6x + y - 2z = 5$$

intersect.

7. Suppose that

$$f(x, y) = e^x \cos \pi y$$

and

$$x = u^2 - v^2, \qquad y = u^2 + v^2.$$

Using the chain rule, compute the values of

$$\frac{\partial f}{\partial u}, \qquad \frac{\partial f}{\partial v}$$

at the point $(u, v) = (1, 1)$.

8. Let

$$f(x, y, z) = \ln\left(x^2 + y^2 - 1\right) + y + 6z.$$

In what direction $\vec{u}$ is $f(x, y, z)$ increasing most rapidly at the point $(1, 1, 0)$? Give your answer as a unit vector $\vec{u}$. What is the directional derivative of f in the direction $\vec{u}$?

9. Find the equation of the tangent plane to the surface

$$xyz = 2$$

at the point $(1, 1, 2)$.

10. Find all critical points of the function

$$f(x, y) = x^4 - 8x^2 + 3y^2 - 6y.$$

and classify them as maximums, minimums, or saddle-point.

11. Let

$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$

be the unit disc and

$$f(x, y) = x^2 - 2x + y^2 + 2y + 1.$$

Find the global maximum and minimum of

$$f : D \to \mathbb{R}$$

At what points (x, y) in D does f attain its maximum and minimum?

12. Suppose that the material for the top and bottom of a rectangular box costs a dollars per square meter and the material for the four sides costs b dollars per square meter. Use the method of Lagrange multipliers to find the dimensions of a box of volume V cubic meters that minimizes the cost of the materials used to construct it. What is the minimal cost?