CALCULUS: **Math 21C, Fall 2010**
Solutions to Sample Questions: Midterm 2

1. Determine the interval of convergence (including the endpoints) for the following power series. State explicitly for what values of x the series converges absolutely, converges conditionally, and diverges. In each case, specify the radius of convergence R and the center of the interval of convergence a.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n}(x-1)^n; \quad \text{(b)} \sum_{n=0}^{\infty} \frac{1}{3^n+1} x^{2n}; \quad \text{(c)} \sum_{n=0}^{\infty} \frac{1}{n^2 5^n}(2x+1)^n.$$

Solution.

- We give two different (but related) methods for finding the solution. Either one is fine.
- **Method I.** Write the power series as $\sum_{n=0}^{\infty} a_n$.
- (a) We have

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^{n+1}(x-1)^{n+1}/(n+1)}{(-1)^n 2^n (x-1)^n/n}\right| \\
&= \lim_{n\to\infty}\left|\frac{2(x-1)n}{n+1}\right| \\
&= 2|x-1| \lim_{n\to\infty}\frac{n}{n+1} \\
&= 2|x-1| \lim_{n\to\infty}\frac{1}{1+1/n} \\
&= 2|x-1|.
\end{aligned}$$

  By the ratio test, the series converges absolutely when

$$2|x-1| < 1$$

  or $|x-1| < 1/2$.

- Therefore, $a = 1$ and $R = 1/2$. The series converges absolutely if $1/2 < x < 3/2$ and diverges if $-\infty < x < 1/2$ or $3/2 < x < \infty$.

- At the endpoint $x = 3/2$ of the interval of convergence, the series becomes
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$
This is the alternating harmonic series. It converges by the alternating series test but does not converge absolutely since the harmonic series diverges. Therefore, the series converges conditionally at $x = 3/2$.

- At the endpoint $x = 1/2$ of the interval of convergence, the series becomes
$$-\sum_{n=1}^{\infty} \frac{1}{n},$$
which a harmonic series. Therefore, the series diverges at $x = 1/2$.

- (b) We have
$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{x^{2(n+1)}/(3^{n+1}+1)}{x^{2n}/(3^n+1)} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^2(3^n+1)}{3^{n+1}+1} \right| \\
&= x^2 \lim_{n\to\infty} \frac{3^n+1}{3^{n+1}+1} \\
&= x^2 \lim_{n\to\infty} \frac{1+1/3^n}{3+1/3^n} \\
&= \frac{x^2}{3}.
\end{aligned}$$
By the ratio test, the series converges absolutely when
$$\frac{x^2}{3} < 1$$
or $|x| < \sqrt{3}$.

- Therefore, $a = 0$ and $R = \sqrt{3}$. The series converges absolutely if $|x| < \sqrt{3}$ and diverges if $|x| > \sqrt{3}$.

- At the endpoints $x = \pm\sqrt{3}$ of the interval of convergence, the series becomes
$$\sum_{n=0}^{\infty} \frac{3^n}{3^n + 1}.$$
Since
$$\lim_{n\to\infty} \frac{3^n}{3^n + 1} = 1,$$
this series diverges by the nth term test.

- (c) We have
$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(2x+1)^{n+1}/[(n+1)^2 5^{n+1}]}{(2x+1)^n/[n^2 5^n]} \right| \\
&= \lim_{n\to\infty} \left| \frac{(2x+1)n^2}{5(n+1)^2} \right| \\
&= \frac{|2x+1|}{5} \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \\
&= \frac{|2x+1|}{5} \lim_{n\to\infty} \frac{1}{(1+1/n)^2} \\
&= \frac{|2x+1|}{5}.
\end{aligned}$$
By the ratio test, the series converges absolutely when
$$\frac{|2x+1|}{5} < 1$$
or $|x + 1/2| < 5/2$.

- Therefore, $a = -1/2$ and $R = 5/2$. The series converges absolutely if $-3 < x < 2$ and diverges if $-\infty < x < -3$ or $2 < x < \infty$.

- At the endpoints $x = -3$, $x = 2$ of the interval of convergence, the series become
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$
These are absolutely convergent p-series, so the series converges absolutely at $x = -3, 2$.

- **Method II.** Write the series as $\sum_{n=0}^{\infty} c_n(x-a)^n$ and use the ratio-test formula for the radius of convergence R:

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

provided that this limit exists. (The rest of the solution is the same as before.)

- (a) We have $a = 1$ and

$$c_n = \frac{(-1)^n 2^n}{n}$$

for $n \geq 1$. Hence

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}2^{n+1}/(n+1)}{(-1)^n 2^n/n} \right| = 2$$

so $R = 1/2$.

- (b) We have $a = 0$. Writing $u = x^2$ the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{3^n+1} u^n$$

The radius of convergence S of this series in u is given by

$$\frac{1}{S} = \lim_{n\to\infty} \left| \frac{1/[3^{n+1}+1]}{1/[3^n+1]} \right| = \frac{1}{3}$$

so the series converges for $|u| < 3$ or $|x| < \sqrt{3}$, and $R = \sqrt{3}$.

- (c) We write the series in standard form as

$$\sum_{n=0}^{\infty} \frac{2^n}{n^2 5^n} \left(x + \frac{1}{2} \right)^n .$$

Thus, $a = -1/2$ and

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{2^{n+1}/[(n+1)^2 5^{n+1}]}{2^n/[n^2 5^n]} \right| = \frac{2}{5}$$

so $R = 5/2$.

2. Let

$$\vec{u} = 2\vec{i} - \vec{j} + 3\vec{k}, \quad \vec{v} = -\vec{i} + 2\vec{j} + 2\vec{k}.$$

Compute: (a) $|\vec{u}|$; (b) $|\vec{v}|$; (c) the angle θ between $\vec{u}$, $\vec{v}$ (you can express it as an inverse trigonometric function); (d) the projection $\mathrm{proj}_{\vec{v}}\vec{u}$ of $\vec{u}$ in the direction of $\vec{v}$.

Solution.

- (a) $|\vec{u}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$.
- (b) $|\vec{v}| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3$.
- (c) We have

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|},$$

and

$$\vec{u} \cdot \vec{v} = 2 \cdot (-1) + (-1) \cdot 2 + 3 \cdot 2 = 2.$$

Hence $\cos\theta = 2/(3\sqrt{14})$ and

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{14}}\right).$$

- (d) We have

$$\mathrm{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\vec{v}.$$

Hence

$$\mathrm{proj}_{\vec{v}}\,\vec{u} = \frac{2}{9}\left(-\vec{i} + 2\vec{j} + 2\vec{k}\right) = -\frac{2}{9}\vec{i} + \frac{4}{9}\vec{j} + \frac{4}{9}\vec{k}.$$

3. (a) Find the area of the triangle with vertices $P(-2, 2, 0)$, $Q(0, 1, -1)$ and $R(-1, 2, -2)$.
(b) Find a parametric equation for the line in which the planes $3x-6y-4z = 15$ and $6x + y - 2z = 5$ intersect.

Solution.

- (a) The area A of the triangle is half the area of the parallelogram spanned by the vectors given by any two sides of the triangle, and the area of the parallelogram is the magnitude of the cross product of the vectors. Hence,
$$A = \frac{1}{2}\left|\vec{PQ} \times \vec{PR}\right|.$$
(The use of any other pair of vectors would be fine and give the same answer.)

- We have
$$\begin{aligned}\vec{PQ} &= (0-(-2))\vec{i} + (1-2)\vec{j} + (-1-0)\vec{k} = 2\vec{i} - \vec{j} - \vec{k},\\ \vec{PR} &= (-1-(-2))\vec{i} + (2-2)\vec{j} + (-2-0)\vec{k} = \vec{i} - 2\vec{k},\end{aligned}$$
and
$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix} = 2\vec{i} + 3\vec{j} + \vec{k}.$$
Hence,
$$A = \frac{1}{2}\sqrt{2^2 + 3^2 + 1^2} = \frac{\sqrt{14}}{2}.$$

- (b) Normal vectors to the planes are
$$\vec{n}_1 = 3\vec{i} - 6\vec{j} - 4\vec{k}, \qquad \vec{n}_2 = 6\vec{i} + \vec{j} - 2\vec{k}.$$
(We read off the components of the normal vector from the coefficients of x, y, z in the Cartesian equation.) A direction vector of the line is therefore
$$\vec{u} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -6 & -4 \\ 6 & 1 & -2 \end{vmatrix} = 16\vec{i} - 18\vec{j} + 39\vec{k}.$$

- To find a point on the line, we look for the coordinates $(x_0, y_0, 0)$ of the point on the intersection of the planes with $z = 0$, say. (This isn't the only way to find a point on the line — any point on the line will do.) Then

$$3x_0 - 6y_0 = 15, \qquad 6x_0 + y_0 = 5.$$

The solution of this linear system is $x_0 = 15/13$, $y_0 = -25/13$. Thus the position vector of a point on the line is

$$\vec{r}_0 = \frac{15}{13}\vec{i} - \frac{25}{13}\vec{j}$$

then the vector form of a parametric equation for the line is

$$\vec{r}(t) = \frac{15}{13}\vec{i} - \frac{25}{13}\vec{j} + t\left(16\vec{i} - 18\vec{j} + 39\vec{k}\right).$$

- The coordinate form of the parametric equation is

$$x = \frac{15}{13} + 16t, \quad y = -\frac{25}{13} - 18t, \quad z = 39t.$$

4. Let

$$f(x,y) = e^{xy}\ln(y).$$

Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy} and f_{yy}. (You do NOT need to simplify your answers.)

Solution.

- The first-order derivatives are

$$\begin{aligned} f_x &= ye^{xy}\ln y, \\ f_y &= xe^{xy}\ln y + \frac{e^{xy}}{y}. \end{aligned}$$

- The second-order derivatives are

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}\left(ye^{xy}\ln y\right) = y^2 e^{xy}\ln y, \\ f_{xy} &= \frac{\partial}{\partial y}\left(ye^{xy}\ln y\right) = e^{xy}\ln y + xye^{xy}\ln y + \frac{ye^{xy}}{y}, \\ f_{yy} &= \frac{\partial}{\partial y}\left(xe^{xy}\ln y + \frac{e^{xy}}{y}\right) \\ &= x^2 e^{xy}\ln y + \frac{xe^{xy}}{y} + \frac{xe^{xy}}{y} - \frac{e^{xy}}{y^2}. \end{aligned}$$

- Alternatively, by the equality of mixed partial derivatives $f_{xy} = f_{yx}$, we could compute f_{xy} from

$$f_{xy} = \frac{\partial}{\partial x}\left(xe^{xy}\ln y + \frac{e^{xy}}{y}\right) = e^{xy}\ln y + xye^{xy}\ln y + \frac{ye^{xy}}{y}.$$

5.(a) Find the Taylor polynomial $P_4(x)$ of order 4 centered at $x = 0$ for the function $f(x) = e^{-x^2}$.
(b) Use Taylor's theorem with remainder to estimate the maximum error $|f(x) - P_4(x)|$ for $0 \le x \le 1$.
(c) Use the results of (a) and (b) to obtain an approximate value for the integral

$$\int_0^1 e^{-x^2}\,dx$$

and estimate the maximum error in your approximate value for the integral.

Solution.

- (a) We have the Taylor series

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots.$$

Replacing x by $-x^2$, we get

$$e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \dots.$$

Thus,

$$P_4(x) = 1 - x^2 + \frac{1}{2}x^4.$$

This is also the Taylor polynomial $P_5(x)$ of order 5 for f, since the coefficient of x^5 is zero.

- Alternatively, we can find $P_4(x)$ by computing the first four derivatives of $f(x)$ at $x = 0$

$$\begin{aligned}
f'(x) &= -2xe^{-x^2}, \\
f''(x) &= (-2x)(-2x)e^{-x^2} - 2e^{-x^2} \\
&= \left(4x^2 - 2\right) e^{-x^2}, \\
f'''(x) &= (-2x)\left(4x^2 - 2\right) e^{-x^2} + 8xe^{-x^2} \\
&= \left(-8x^3 + 12x\right) e^{-x^2}, \\
f^{(4)}(x) &= (-2x)\left(-8x^3 + 12x\right) e^{-x^2} + \left(-24x^2 + 12\right) e^{-x^2} \\
&= \left(16x^4 - 48x^2 + 12\right) e^{-x^2},
\end{aligned}$$

and using the formula for Taylor coefficients,

$$\begin{aligned}
c_0 &= f(0) = 1,\\
c_1 &= f'(0) = 0,\\
c_2 &= \frac{f''(0)}{2!} = \frac{-2}{2} = -1,\\
c_3 &= \frac{f'''(0)}{3!} = 0,\\
c_4 &= \frac{f^{(4)}(0)}{4!} = \frac{12}{24} = \frac{1}{2},
\end{aligned}$$

but this involves a lot more algebra than the previous method.

- (b) By Taylor's theorem with remainder, if $0 \le x \le 1$, then

  $$e^{-x^2} = P_4(x) + R_4(x)$$

  where

  $$R_4(x) = \frac{f^{(5)}(c)x^5}{5!}$$

  for some $0 < c < x$.

- Since $0 \le x \le 1$, we have $0 < c < 1$. If

  $$\left|f^{(5)}(c)\right| \le M$$

  for $0 < c < 1$, it follows that for $0 \le x \le 1$

  $$\left|e^{-x^2} - P_4(x)\right| = |R_4(x)| \le \frac{Mx^5}{5!}.$$

- Unfortunately, it looks like we have to calculate the fifth-order derivative of f to find an estimate for M and the algebra becomes much more complicated than I expected, so the details are omitted (and not required). A simpler way to estimate the error (based on integrating the whole Taylor series term-by-term instead of using the remainder) is described at the end.

- (c) We have approximately that

  $$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 P_4(x)\,dx,$$

and

$$\int_0^1 P_4(x)\,dx = \int_0^1 \left(1 - x^2 + \frac{1}{2}x^4\right)\,dx$$
$$= \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5\right]_0^1$$
$$= 1 - \frac{1}{3} + \frac{1}{10}$$
$$= \frac{23}{30}.$$

- We can estimate the error in this approximate value of the integral in terms of the constant M from (b) as follows:

$$\left|\int_0^1 e^{-x^2}\,dx - \frac{23}{30}\right| = \left|\int_0^1 e^{-x^2}\,dx - \int_0^1 P_4(x)\,dx\right|$$
$$= \left|\int_0^1 \left[e^{-x^2} - P_4(x)\right]\,dx\right|$$
$$\le \int_0^1 \left|e^{-x^2} - P_4(x)\right|\,dx$$
$$\le \int_0^1 \frac{Mx^5}{5!}\,dx$$
$$\le \frac{M}{6!}.$$

- To get an explicit estimate for the error by a simpler method, note that by replacing x in the Taylor series for e^x by $-x^2$, we get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

where this series converges for all x (because the Taylor series for e^x converges for all x). Since we can integrate Taylor series term by term within their interval of convergence, we have

$$\int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}.$$

Thus, we have an exact expression for the integral as the sum of an infinite series:

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{1\cdot 3} + \frac{1}{2!\cdot 5} - \frac{1}{3!\cdot 7} + \frac{1}{4!\cdot 9} - \dots + \frac{(-1)^n}{n!(2n+1)} + \dots.$$

This is an alternating series whose partial sums are alternately larger and smaller than the sum of the series, so

$$1 - \frac{1}{1\cdot 3} + \frac{1}{2!\cdot 5} - \frac{1}{3!\cdot 7} \le \int_0^1 e^{-x^2}\,dx \le 1 - \frac{1}{1\cdot 3} + \frac{1}{2!\cdot 5},$$

or

$$\frac{26}{35} \le \int_0^1 e^{-x^2}\,dx \le \frac{23}{30}.$$

- To four significant figures, the numerical values of the integrals are

$$\int_0^1 P_4(x)\,dx = 0.7667, \qquad \int_0^1 e^{-x^2}\,dx = 0.7468.$$

  The actual value is less that the approximate value, and $26/35 \approx 0.7429$, so the lower bound on the approximation is also correct. (The lower bound gives a more accurate approximation because we obtained it by using more terms in the series than the upper bound.)

- The integral of e^{-x^2} cannot be expressed in terms of elementary functions, but it can be expressed in terms of the error function erf as

$$\int_0^1 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\operatorname{erf}(1).$$

6. Suppose that the functions $f(x)$, $g(x)$ have the Taylor series expansions at zero, up to second degree terms, given by

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots, \qquad g(x) = b_0 + b_1 x + b_2 x^2 + \dots$$

(a) According to Taylor's theorem, how are a_0, a_1, a_2 given in terms of f and its derivatives and b_0, b_1, b_2 in terms of g and its derivative?

(b) Find the Taylor series for $h(x) = f(x)g(x)$ at zero, up to second degree terms, by multiplying the Taylor series for $f(x)$ and $g(x)$.

(c) Use the product rule to compute $h'(x)$, $h''(x)$ in terms of the derivatives of $f(x)$, $g(x)$. Show that the use of these expressions in Taylor's theorem for $h(x)$ gives the same series as the one you found in (b).

Solution.

- (a) By Taylor's theorem

$$\begin{aligned} a_0 &= f(0), \quad a_1 = f'(0), \quad a_2 = \frac{f''(0)}{2!}, \quad \dots, \\ b_0 &= g(0), \quad b_1 = g'(0), \quad b_2 = \frac{g''(0)}{2!}, \quad \dots. \end{aligned}$$

- (b) Multiplying the Taylor series of f and g, we get

$$\begin{aligned} h(x) &= \left(a_0 + a_1 x + a_2 x^2 + \dots\right)\left(b_0 + b_1 x + b_2 x^2 + \dots\right) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)\, x + (a_0 b_2 + a_1 b_1 + a_2 b_0)\, x^2 + \dots. \end{aligned}$$

  Thus,

$$h(x) = c_0 + c_1 x + c_2 x^2 + \dots$$

  where

$$c_0 = a_0 b_0, \quad c_1 = a_0 b_1 + a_1 b_0, \quad c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0, \quad \dots.$$

- (c) By the product rule, we have

$$h' = fg' + f'g, \qquad h'' = fg'' + 2f'g' + f''g.$$

The first few Taylor coefficients of h at 0 are therefore

$$\begin{aligned}
c_0 &= h(0) \\
&= f(0)g(0) \\
&= a_0 b_0, \\
c_1 &= h'(0) \\
&= f(0)g'(0) + f'(0)g(0) \\
&= a_0 b_1 + a_1 b_0, \\
c_2 &= \frac{h''(0)}{2!} \\
&= \frac{1}{2}\left[f(0)g''(0) + 2f'(0)g'(0) + f''(0)g(0)\right] \\
&= f(0)\frac{g''(0)}{2!} + f'(0)g'(0) + \frac{f''(0)}{2!}g(0) \\
&= a_0 b_2 + a_1 b_1 + a_2 b_0.
\end{aligned}$$

These expressions agree with what we found by multiplication of the Taylor series.