Math 22B Solutions
Homework 2
Spring 2008

Section 2.1

16. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}$, with $y(\pi) = 0$ and $t = 0$

Solution Let $\mu(t) = e^{\int \frac{2}{t}dt} = e^{t^2}$. If we multiply both sides of the given equation by $\mu(t)$, we get:

$$t^2(y)' + \frac{2}{t} = \cos t$$

$$(t^2 y)' = \cos t$$

Then integrate both sides to get:

$$t^2 y = \sin t + c$$

$$y = \frac{1}{t^2}\sin t + \frac{c}{t^2}$$

Setting initial values we get:

$$y = \frac{1}{\pi^2}\sin \pi + \frac{c}{\pi^2} = 0$$

$$y = 0 + \frac{c}{\pi^2} = 0 \Rightarrow c = 0$$

$$y = \frac{1}{t^2}\sin t$$

22.

(b) $2y' - y = e^{\frac{t}{3}}$, with $y(0) = a$

Solution

$$\mu t = e^{\frac{-t}{2}}$$

$$2e^{\frac{-t}{2}}y' - e^{\frac{-t}{2}}y = e^{\frac{t}{3}-\frac{t}{2}}$$

$$y(t) = -3e^{\frac{t}{3}} + ce^{\frac{t}{2}}$$

$$y(0) = -3 + c = a \Rightarrow c = a + 3$$
$$y(t) = -3e^{\frac{t}{3}} + (a+3)e^{\frac{t}{2}}$$

Differentiate to find critical points:

$$y'(t) = -e^{\frac{t}{3}} + \frac{(a+3)}{2}e^{\frac{t}{2}}$$

$$y'(0) = -1 + \frac{(a+3)}{2} = \frac{(a+1)}{2},\ a_0 = -1 \textit{ is a critical value.}$$

(c) Solution For $g = -1$, $y(t) = -3e^{\frac{t}{2}} + 2e^{\frac{t}{2}}$. This is dominated by $e^{\frac{t}{2}}$.

30. $y' - y = 1 + 3\sin t$

Solution

$$\mu t = e^{-t}$$
$$(e^{-t}y)' = e^{-t} + 3e^{-t}\sin t$$
$$e^{-t}y = -e^{-t} + \frac{3}{2}(-e^{-t}\cos t - e^{-t}\sin t) + c$$
$$y = -1 - \frac{3}{2}e^{-t}(\cos t + \sin t) + ce^{t}$$

If $y(t)$ is bounded, $c = 0$, and

$$y(0) = -1 - \frac{3}{2} + c = y_0$$

Where $c = y_0 + \frac{5}{2} = 0$ and $y_0 = -\frac{5}{2}$.

Section 2.2

7. Solution The equation $\frac{dy}{dx} = \frac{x-e^{-x}}{y+e^{y}}$ is separable, so

$$(y + e^{y})dy = (x - e^{-x})dx$$
$$\int (y + e^{y})dy = \int (x - e^{-x})dx$$
$$\frac{y^2}{2} + e^{y} = \frac{x^2}{2} + e^{-x} + c$$

Multiply by 2

$$y^2 + 2e^{y} = x^2 + 2e^{-x} + c$$

13.

(a) $y' = \frac{2x}{y+x^2y}$, with $y(0) = -2$

$$y' = \frac{2x}{y(1+x^2)} \ \Rightarrow \ ydy = \frac{2x}{1+x^2}dx$$

$$\int ydy = \int \frac{2x}{1+x^2}dx$$

$$y^2 = \ln 1 + x^2 + c$$

$$y(t) = -\sqrt{2\ln 1 + x^2 + c} \ \Rightarrow \ y(0) = \sqrt{2(0) + c} = -2, \ where \ c = 4.$$

(b) See figure sheet

(c) The solution is defined as long as $\ln 1 + x^2$ exists. So we must have $0 < x$.

30.

(a)

$$\frac{dy}{dx} = \frac{y - 4x}{x - y} = \frac{y - 4x}{(1 - \frac{y}{x})x} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$$

(b) Let $v = \frac{y}{x}$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(c)

$$v + x\frac{dv}{dx} = \frac{v - 4}{1 - v}$$

$$x\frac{dv}{dx} = \frac{v - 4}{1 - v} - v = \frac{v - 4 - v(1 - v)}{1 - v}$$

$$= \frac{v - 4 - v + v^2}{1 - v} = \frac{v^2 - 4}{1 - v}$$

(d) $\frac{1-v}{v^2-4}dv = \frac{1}{x}dx$ Use partial fractions to integrate the left hand side:

$$\frac{1 - v}{v^2 - 4} = \frac{A}{v - 2} + \frac{B}{v + 2}$$

$$Av + 2A + Bv - 2B$$

$$A + B = -1$$

$$2A - 2B = 1$$

$$4A = -1 \;\Rightarrow\; A = -\frac{1}{4},\; B = -\frac{3}{4}$$

$$\int \frac{-1}{4(v-2)} - \frac{3}{4(v+2)} dv = \int \frac{1}{x} dx$$

$$\frac{-1}{4} \ln|v-2| - \frac{3}{4} \ln|v+2| = \ln x - c$$

$$\ln|v-2||v+2|^3|x^4| = c$$

$$|v-2||v+2|^3|x^4| = c$$

(e)

$$\frac{y}{x} - 2||\frac{y}{x} + 2|^3|x^4| = c$$

Section 2.4

6. Determine, without solving, an interval in which the solution of the intitial value problem $\ln t y' + y = \cot t$, where $y(2) = 3$ is certain to exist.

Solution Notice that $\ln t$ is continuous on the interval $(0, \infty)$, and $\cot t$ is continuous on the interval $(-\infty, \infty)$. Thus, by theorem 2.4.7, the interval we are looking for is $(0, \infty)$.

9. State where in the ty-plane the hypotheses of theorem 2.4.2 are satisfied for $y' = \frac{\ln|ty|}{1-t^2+y^2}$.

Solution We want $\frac{\ln|ty|}{1-t^2+y^2}$ and $\frac{\partial f}{\partial y}$. Well,

$$\frac{\partial f}{\partial y} = \frac{(1-t^2+y^2)\frac{t}{y} - \ln ty2y}{(1-t^2+y^2)^2}$$

$$= \frac{t}{y(1-t^2+y^2)} - \frac{2y\ln|ty|}{(1-t^2+y^2)}$$

The function $f(t, y)$ is discontinuous when $1-t^2+y^2 = 0$, or when $1 = t^2-y^2$; also when $|ty| = 0 \;\Rightarrow\; t = 0$, or $y = 0$. It is clear that $\frac{\partial f}{\partial y}$ has the same points of discontinuity. So thm. 2.4.2 is satisfied anywhere except on the coordinate axes and on the hyperbola $t^2 - y^2 = 1$.

14. Solve the initial value problem $y' = 2ty^2$, with $y(0) = y_0$, and determine how the interval in which the soln. exists depends on the initial value y_0.

Solution This equation is separable, so we separate and integrate:

$$\int \frac{dy}{y^2} = \int 2tdt$$

$$\frac{-1}{y} = t^2 + c$$

$$y = \frac{-1}{t^2 + c}$$

$$y(0) = \frac{-1}{c} = y_0 \ \Rightarrow \ c = \frac{-1}{y_0}$$

$$\Rightarrow y = \frac{-1}{t^2 - \frac{1}{y_0}} = ... = \frac{y_0}{1 - y_0 t^2}$$

For $y_0 > 0$, the soln. exists as long as $t^2 < \frac{1}{y_0}$. For $y \leq 0$, the soln. is defined for all t.

21. (a) From example 3, the solns. are of the form

$$y = \chi(t) = \begin{cases} 0 & \text{if } 0 \leq t < t_0 \\ \pm[\frac{2}{3}(t - t_0)]^{\frac{3}{2}} & \text{if } t \geq t_0 \end{cases}$$

Substituting $t = 1$ into the soln., we now want to find a $t_0 > 0$ such that $1 = \pm[\frac{2}{3}(t - t_0)]^{\frac{3}{2}}$. Solving this equation for t_0, we get:

$$\pm(\frac{3}{2})^{\frac{3}{2}} = (1 - t_0)^{\frac{3}{2}}$$

$$\frac{3}{2} = 1 - t_0$$

$$\frac{1}{2} = -t_0$$

$$t_0 = \frac{-1}{2}$$

But this is a contradiction since $t_0 > 0$. So no soln. goes through $(1, 1)$.

(b) Again, we substitute in $t = 2$ and solve for t_0. We get:

$$1 = \pm[\frac{2}{3}(2 - t_0)]^{\frac{3}{2}}$$

$$1 = \frac{2}{3}(2 - t_0)$$
$$\frac{3}{2} = 2 - t_0$$
$$\frac{-1}{2} = -t_0 \;\Rightarrow\; \frac{1}{2} = t_0$$

Hence, there is a soln. that passes through $(2, 1)$.

(c) Clearly we have $y = 0$ for any $t_0 > 2$. For $0 < t \leq 2$, we get an upper bound on y by letting $t_0 = 0$. Then we have:

$$y < [\frac{2}{3}2]^{\frac{3}{2}} = (\frac{4}{3})^{\frac{3}{2}}$$

For a lower bound on y, we consider the negative of this case. Hence, $-(\frac{4}{3})^{\frac{3}{2}} < y < (\frac{4}{3})^{\frac{3}{2}}$

26. (a) We have $y = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)g(s)ds + \frac{c}{\mu(t)}$, so clearly $y_1(t) = \frac{1}{\mu(t)}$ and $y_2(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)g(s)ds$.

(b) We have $y_1(t) = \frac{1}{\mu(t)}$, and $\frac{1}{\mu(t)} = exp(-\int p(t)dt)$ by definition. Hence, $y_1'(t) = -p(t)\frac{1}{\mu(t)} = -p(t)y_1$. That is, $y_1'(t) + p(t)y_1 = 0$.

(c) We have

$$y_2' = (-p(t)\frac{1}{\mu(t)}) \int_0^t \mu(t)g(s)ds + \frac{1}{\mu(t)}\mu(t)g(t) = -p(t)y_2 + g(t)$$

That is, $y_2' + p(t)y_2 = g(t)$, as desired.