Ordinary Differential Equations

Math 22B-002, Spring 2008

Final Exam: Solutions

NAME...

SIGNATURE..

I.D. NUMBER..

No books, notes, or calculators. Show all your work

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total	200	

1. [20 pts.] (a) Find the solution of the initial value problem

$$ty' = \frac{1}{y+1}, \qquad y(1) = 0.$$

(b) For what t-interval is the solution defined?

Solution.

- (a) The equation is separable. Separating variables, we get

  $$\int (y+1)\,dy = \int \frac{dt}{t}.$$

  Evaluation of the integrals gives

  $$\frac{1}{2}(y+1)^2 = \ln t + C.$$

  Imposing the initial condition, and using $\ln 1 = 0$, we get

  $$\frac{1}{2} = C.$$

  Solving for y, we find that

  $$y(t) = \sqrt{2\ln t + 1} - 1.$$

- Instead, you can integrate the separated equation as

  $$\frac{1}{2}y^2 + y = \ln t + C,$$

  which just differs from the previous solution in how the constant of integration is chosen. Imposing the initial condition, we get $C = 0$, so

  $$y^2 + 2y - 2\ln t = 0.$$

  Solving this quadratic equation for y, we get

  $$y(t) = \frac{-2 \pm \sqrt{4 + 8\ln t}}{2}.$$

  Choosing the +-root, which gives $y = 0$ when $t = 1$, and simplifying the result, we get

  $$y(t) = \sqrt{2\ln t + 1} - 1$$

  as before.

- (b) This solution is defined and continuously differentiable provided that $2 \ln t + 1 > 0$, or $\ln t > -1/2$, which holds in the interval

$$e^{-1/2} < t < \infty.$$

2. [20 pts.] Suppose that a is a constant, and consider the initial value problem

$$y' - y = e^{at}, \qquad y(0) = 0.$$

(a) Find the solution if $a \neq 1$.

(b) Find the solution if $a = 1$.

(c) Show that the solution in (b) is the limit of the solution in (a) as $a \to 1$. (Hint: use l'Hospital's rule.)

Solutions.

- The equation is linear and nonhomogeneous, so we use the integrating factor method. An integrating factor is

  $$\mu(t) = \exp \int (-1)\, dt = e^{-t}.$$

  Multiplying the equation by e^{-t} and rewriting the left-hand side as an exact derivative, we get

  $$\left(e^{-t} y\right)' = e^{(a-1)t}. \tag{1}$$

- (a) If $a \neq 1$, then an integration of this equation gives

  $$e^{-t} y = \frac{1}{a-1}\, e^{(a-1)t} + C.$$

  Imposition of the initial condition gives

  $$C = -\frac{1}{a-1}.$$

  The solution of the IVP is therefore

  $$y(t) = \frac{e^{at} - e^{t}}{a-1}.$$

- If $a = 1$, equation (1) becomes

  $$\left(e^{-t} y\right)' = 1.$$

  Integration of this equation gives

  $$e^{-t} y = t + C.$$

Imposition of the initial condition gives

$$C = 0.$$

The solution is therefore

$$y(t) = te^t.$$

- (c) By l'Hospital's rule the limit of the solution in (a) as $a \to 1$ is given by

$$\begin{aligned} \lim_{a\to 1} y(t,a) &= \lim_{a\to 1} \frac{e^{at} - e^t}{a-1} \\ &= \lim_{a\to 1} \frac{\frac{d}{da}\left[e^{at} - e^t\right]}{\frac{d}{da}[a-1]} \\ &= \lim_{a\to 1} \frac{te^{at}}{1} \\ &= te^t. \end{aligned}$$

Thus, we obtain the solution in (b).

Remark. This problem is an example of how the 'resonant' solution te^t for a nonhomogeneous term that solves the homogeneous equation arises as a limit of the nonresonant solutions.

3. [20 pts.] Find the general solutions of the following ODEs.
(a) $y'' - 4y' + 5y = 0$.
(b) $y'' + 3y' - 4y = 0$.

Solution.

- These are linear second order, homogeneous ODEs with constant coefficients, so we can solve them by finding the the roots of the associated characteristic equation.

- (a) The characteristic equation is

  $$r^2 - 4r + 5 = 0.$$

  By the quadratic formula, the solutions are

  $$r = 2 \pm i.$$

  Hence, the general solution of the ODE is

  $$y(t) = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t.$$

- (b) The characteristic equation is

  $$r^2 + 3r - 4 = 0.$$

  The solutions are

  $$r = -4, 1.$$

  Hence, the general solution of the ODE is

  $$y(t) = c_1 e^{-4t} + c_2 e^t.$$

4. [20 pts.] Find particular solutions of the following ODEs.
(a) $y'' - y' + 3y = \sin t$.
(b) $y'' + 2y' - 3y = e^t$.

Solution.

- (a) If we include a term proportional to $\sin t$ in $y(t)$, then y' introduces a term proportional to $\cos t$ in the equation. We therefore try a particular solution of the form

$$y(t) = A\cos t + B\sin t.$$

We compute that

$$y'' - y' + 3y = (-A - B + 3A)\cos t + (-B + A + 3B)\sin t.$$

Hence we get a solution if

$$2A - B = 0, \qquad A + 2B = 1.$$

This system has the solution $A = 1/5$, $B = 2/5$, so a particular solution of the ODE is

$$y(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t.$$

(b) The nonhomogeneous term e^t is a solution of the associated homogeneous ODE. We therefore look for a particular solution of the form

$$y(t) = Ate^t.$$

We compute that

$$y'' + 2y' - 3y = Ate^t + 2Ae^t + 2A\left(te^t + e^t\right) - 3Ate^t = 4Ae^t.$$

Hence, we get a solution if $A = 1/4$, and a particular solution is

$$y(t) = \frac{1}{4}te^t.$$

5. [20 pts.] Suppose that the coefficient functions $p(t)$, $q(t)$ are continuous in the interval $0 < t < \pi$, and the functions $y_1(t) = t$, $y_2(t) = \sin t$ are solutions of the ODE

$$y'' + p(t)y' + q(t)y = 0 \qquad 0 < t < \pi.$$

(a) Compute the Wronskian of y_1, y_2. Are they linearly independent on the interval $0 < t < \pi$? Is the pair $\{y_1, y_2\}$ a fundamental set of solutions for the ODE? Could $p(t)$, $q(t)$ be continuous on $-\pi < t < \pi$? Explain your answers.

(b) Find the solution $y(t)$ of the initial value problem for the ODE with initial conditions

$$y\left(\frac{\pi}{2}\right) = 0, \qquad y'\left(\frac{\pi}{2}\right) = 2.$$

Solution.

- (a) The Wronskian $W(t)$ is

$$W(t) = \begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix} = t\cos t - \sin t.$$

- The Wronskian is not identically zero; for example, $W(\pi/2) = -1 \neq 0$. Therefore the functions are linearly independent.

- The functions form a fundamental set of solutions of the ODE, since this is true of any two linearly independent solutions of a second order, linear, homogeneous, scalar ODE.

- The functions $p(t)$, $q(t)$ cannot be continuous on $-\pi < t < \pi$. If they were, Abel's theorem would imply that the Wronskian of any pair of solutions is either always zero or never zero in that interval, but $W(0) = 0$ and $W(\pi/2) \neq 0$.

- (b) By the superposition principle for linear, homogeneous ODEs,

$$y(t) = c_1 t + c_2 \sin t$$

is a solution for arbitrary constant c_1, c_2. We have

$$y'(t) = c_1 + c_2 \cos t,$$

and imposing the initial conditions, we get

$$\frac{\pi}{2}c_1 + c_2 = 0, \qquad c_1 = 2.$$

Hence, $c_1 = 2$, $c_2 = -\pi$, and the solution is

$$y(t) = 2t - \pi \sin t.$$

Remark. As you can verify directly, the ODE with these solutions is

$$y'' - \left(\frac{t\tan t}{\tan t - t}\right) y' + \left(\frac{\tan t}{\tan t - t}\right) y = 0.$$

The coefficient functions are discontinuous at $t = 0$, since

$$\lim_{t\to 0}\left(\frac{\tan t}{\tan t - t}\right) = +\infty, \qquad \lim_{t\to 0^{\pm}}\left(\frac{t\tan t}{\tan t - t}\right) = \mp\infty.$$

6. [20 pts.] The displacement $y(t)$ of an undamped oscillator of mass $m > 0$ on a spring with spring constant $k > 0$, and initial displacement $a \neq 0$ and initial velocity 0 satisfies

$$my'' + ky = 0, \qquad y(0) = a, \quad y'(0) = 0.$$

(a) Solve this initial value problem.

(b) Show that the solution is periodic with period T, meaning that $y(t+T) = y(t)$, and express T in terms of m and k.

(c) For what times t does the oscillator pass through equilibrium, meaning that $y(t) = 0$?

Solution.

- (a) We write the ODE as

  $$y'' + \omega_0^2 y = 0$$

  where

  $$\omega_0 = \sqrt{\frac{k}{m}}.$$

  The characteristic equation is $r^2 + \omega_0^2 = 0$ with roots $r = \pm i\omega_0$, so the general solution of the ODE is

  $$y(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

- Imposing the initial conditions, we get

  $$a = c_1, \qquad \omega_0 c_2 = 0,$$

  so $c_1 = a$, $c_2 = 0$ and the solution is

  $$y(t) = a \cos \omega_0 t.$$

- (b) Let

  $$T = \frac{2\pi}{\omega_0}.$$

Then, since $\cos x$ is 2π-periodic in x,

$$\begin{aligned} y(t+T) &= a\cos\left[\omega_0\left(t+T\right)\right] \\ &= a\cos\left(\omega_0 t + 2\pi\right) \\ &= a\cos\left(\omega_0 t\right) \\ &= y(t). \end{aligned}$$

Thus, the solution is periodic with period T. Using the expression for ω_0, we find that the period is given in terms of m and k by

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

(c) We have $y(t) = 0$ if

$$\cos\omega_0 t = 0,$$

which implies that

$$\omega_0 t = \frac{\pi}{2} + n\pi$$

where $n = 0 \pm 1, \pm 2, \ldots$ is any integer. It follows that

$$t = \frac{\pi}{\omega_0}\left(n + \frac{1}{2}\right) = \left(\frac{2n+1}{4}\right)T.$$

7. [20 pts.] (a) Find the general solution for $\vec{x}(t)$ of the following 2×2 system:

$$\vec{x}' = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} \vec{x}.$$

(b) Classify the equilibrium $\vec{x} = 0$. Is it stable or unstable?

Solution.

- (a) The characteristic polynomial of the matrix

$$A = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix}$$

is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -2-\lambda & 3 \\ 1 & -4-\lambda \end{vmatrix} \\ &= \lambda^2 + 6\lambda + 5 \\ &= (\lambda+1)(\lambda+5). \end{aligned}$$

The eigenvalues of A are therefore $\lambda = -1$, $\lambda = -5$. An a eigenvector $\vec{\xi}$ of A with eigenvalue λ satisfies $(A - \lambda I)\vec{\xi} = 0$.

- If $\lambda = -1$, we get

$$\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0,$$

and an eigenvector is

$$\vec{\xi} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

- If $\lambda = -5$, we get

$$\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0,$$

and an eigenvector is

$$\vec{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- The general solution is

$$\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- (b) The equilibrium $\vec{x} = 0$ is a stable node.

8. [20 pts.] Suppose that a 2×2 matrix A has the following eigenvalues and eigenvectors:

$$r_1 = 2, \quad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad r_2 = 1, \quad \vec{\xi}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Sketch the trajectories of the system $\vec{x}' = A\vec{x}$, where $\vec{x} = (x_1, x_2)^T$, in the phase plane. Classify the equilibrium $\vec{x} = 0$. Is it stable or unstable?

(b) Sketch the graphs of $x_1(t)$ and $x_2(t)$ versus t for the solution that satisfies the initial condition $x_1(0) = 2$, $x_2(0) = 0$.

Solution.

- The equilibrium $\vec{x} = 0$ is an unstable node.
- Sketches are omitted.

9. [20 pts.] (a) Use the definition of the matrix exponential

$$e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \ldots + \frac{1}{n!}t^nA^n + \ldots,$$

to compute e^{tA} for the following 2×2 matrix:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(b) Use your result from (a) to find the solution $\vec{x}(t) = (x_1(t), x_2(t))^T$ of the initial value problem

$$\vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Write out the solutions for the components $x_1(t)$, $x_2(t)$ explicitly.

Solution.

- We compute that

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

  It follows that $A^3 = -A$, $A^4 = I$, $A^5 = A$, $A^6 = -I$, and so on.

- We therefore get

$$\begin{aligned} e^{tA} &= I + tA - \frac{1}{2!}t^2I - \frac{1}{3!}t^3A + \frac{1}{4!}t^4I + \frac{1}{5!}t^5A - \ldots \\ &= \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \ldots\right)I + \left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \ldots\right)A \\ &= \cos t\, I + \sin t\, A, \end{aligned}$$

  where we use the Taylor series expansions

$$\begin{aligned} \cos t &= 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \ldots, \\ \sin t &= t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \ldots. \end{aligned}$$

We therefore have

$$e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

(b) The solution of the ODE is

$$\begin{aligned} \vec{x}(t) &= e^{tA}\,\vec{x}(0) \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos t - \sin t \\ \cos t + \sin t \end{pmatrix}. \end{aligned}$$

In components,

$$x_1(t) = \cos t - \sin t, \qquad x_2(t) = \cos t + \sin t.$$

10. [20 pts.] Suppose that $-1 < a < 1$ is a constant parameter, and $y(t)$ satisfies the ODE

$$y' = (a - y^2)(y - 2).$$

(a) Find the equilibria, sketch the phase line, and determine the stability of the equilibria in each of the following cases: (i) $-1 < a < 0$; (ii) $a = 0$; (iii) $0 < a < 1$.

(b) Suppose that $y(t)$ is the solution of the ODE that satisfies the initial condition $y(0) = 0$. What is the behavior of $y(t)$ as $t \to +\infty$ in each of the cases (i), (ii), (iii).

Solution.

- (a) Let $f(y, a) = (a - y^2)(y - 2)$. The equilibria are solutions of $f(y, a) = 0$, which gives $y = 2$, and $y = \pm\sqrt{a}$ if $a \geq 0$.

- (i) If $-1 < a < 0$, then there is one equilibrium $y = 2$. In this case, we have $f(y, a) > 0$ if $y < 2$ and $f(y, a) < 0$ if $y > 2$. The phase line therefore looks like:

  $$\longrightarrow\longrightarrow\longrightarrow \cdot \longleftarrow$$

  The equilibrium $y = 2$ is asymptotically stable.

- (ii) If $a = 0$, then there are two equilibria, $y = 0$, $y = 2$. In this case, we have $f(y, a) > 0$ if $-\infty < y < 0$ or $0 < y < 2$, and $f(y, a) < 0$ if $y > 2$. The phase line therefore looks like:

  $$\longrightarrow\longrightarrow \cdot \longrightarrow \cdot \longleftarrow$$

  The equilibrium $y = 0$ is semi-stable, and the equilibrium $y = 2$ is asymptotically stable.

- (iii) If $0 < a < 1$, then there are three equilibria, $y = \pm\sqrt{a}$, $y = 2$. In this case, we have $f(y, a) > 0$ if $-\infty < y < -\sqrt{a}$ or $\sqrt{a} < y < 2$, and $f(y, a) < 0$ if $-\sqrt{a} < y < \sqrt{a}$ or $y > 2$. The phase line therefore looks like:

  $$\longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow$$

  The equilibrium $y = -\sqrt{a}$ is asymptotically stable, the equilibrium $y = \sqrt{a}$ is unstable, and the equilibrium $y = 2$ is asymptotically stable.

- (b) Looking at the phase lines, we see that if $y(0) = 0$, then: (i) $y(t) \to 2$ as $t \to +\infty$ if $-1 < a < 0$; (ii) $y(t) = 0$ for all t if $a = 0$; (iii) $y(t) \to -\sqrt{a}$ as $t \to +\infty$ if $0 < a < 1$.

Remark. Note how the asymptotic behavior of the solution jumps discontinuously from the equilibrium $y = 2$ to the equilibrium $y = -\sqrt{a}$, near $y = 0$, as a passes continuously through 0. There is a saddle-node bifurcation at $a = 0$ which leads to a qualitative change in the dynamics of the system.