ORDINARY DIFFERENTIAL EQUATIONS
Math 22B-002, Spring 2007
Sample Final Exam

NAME...

SIGNATURE..

I.D. NUMBER..

No books, notes, or calculators. Show all your work

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total	200	

1. [20 pts.] (a) Solve the following initial value problem for $y(t)$ in $t > 0$:

$$y' - \frac{2}{t} y = t, \qquad y(1) = 2.$$

(b) How does $y(t)$ behave as $t \to 0^+$?

2. [20 pts.] (a) Solve the initial value problem

$$y' + (\cos t)y^2 = 0, \qquad y(0) = y_0,$$

where y_0 is an arbitrary constant.

(b) For what values of the initial data y_0 is your solution defined for all $-\infty < t < +\infty$?

3. [20 pts.] Consider the ordinary differential equation

$$y' = (y-3)\left(y^2-1\right)$$

(a) Sketch a graph of the right-hand side of this equation as a function of y, and find all equilibrium solutions of the equation.

(b) Sketch the phase line of the equation, and determine the stability of the equilibria you found in (a).

(c) How does the solution with $y(0) = 0$ behave as $t \to +\infty$? How does the solution with $y(0) = 2$ behave as $t \to -\infty$?

4. [20 pts.] (a) Find the general solution of the equation

$$y'' + y' - 2y = 0.$$

(b) Find the general solution of the equation

$$y'' - 2y' + y = 0.$$

5. [20 pts.] Consider the nonhomogeneous ordinary differential equation

$$y'' + 4y = 4t^2 + 10e^{-t}.$$

(a) Find a fundamental pair of solutions for the associated homogeneous equation

(b) Find a particular solution of the nonhomogeneous equation.

(c) Write out the general solution of the nonhomogeneous equation

6. [20 pts.] Consider the ordinary differential equation (in $t > 0$)

$$ty'' - (t+1)y' + y = 0.$$

Note that $y_1(t) = e^t$ is a solution of this equation.

(a) Write $y(t) = v(t)e^t$ and derive an equation for $v(t)$.

(b) Solve the equation for $v(t)$ you derived in (a).

(c) Write out an expression for the general solution of the original ordinary differential equation.

7. [20 pts.] Consider a damped simple harmonic oscillator whose displacement $u(t)$ satisfies the ODE

$$mu'' + \gamma u' + ku = 0,$$

where m, γ, k are positive constants. Let

$$E(t) = \frac{1}{2}m\,(u')^2 + \frac{1}{2}ku^2.$$

Show that if $\gamma > 0$ then

$$E'(t) < 0$$

unless $u(t) = 0$. Give a physical interpretation of this result.

8. [20 pts.] Find the general solution (expressed in terms of real-valued functions) of the following 2×2 system

$$\vec{x}'(t) = \begin{pmatrix} -1 & 5 \\ -2 & -3 \end{pmatrix} \vec{x}(t).$$

9. [20 pts.] Suppose that

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$. Compute e^{tA}.

10. [20 pts.] Suppose that the 2×2 matrix A has the following eigenvalues and eigenvectors:

$$r_1 = -3, \quad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad r_2 = -1, \quad \vec{\xi}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Sketch the trajectories of the system $\vec{x}'(t) = A\vec{x}(t)$, where $\vec{x} = (x_1, x_2)^T$.

(b) Sketch the graphs of $x_1(t)$ and $x_2(t)$ versus t for the solution that satisfies the initial condition $x_1(0) = 3$, $x_2(0) = 1$.