Ordinary Differential Equations

Math 22B-002, Spring 2007

Sample Final Exam Solutions

1. (a) Solve the following initial value problem for $y(t)$ in $t > 0$:

$$y' - \frac{2}{t}y = t, \qquad y(1) = 2.$$

(b) How does $y(t)$ behave as $t \to 0^+$?

Solution.

- (a) This is a first-order linear ODE. The integrating factor is $1/t^2$:

$$\begin{aligned} &\frac{1}{t^2}y' - \frac{2}{t^3}y = \frac{1}{t}, \\ &\left(\frac{1}{t^2}y\right)' = \frac{1}{t}, \\ &\frac{1}{t^2}y = \ln t + c, \\ &y = t^2 \ln t + ct^2. \end{aligned}$$

- Imposing the initial condition, we get $c = 2$, so the solution is

$$y(t) = t^2 \ln t + 2t^2.$$

- (b) Since $t \ln t \to 0$ as $t \to 0^+$, $y(t) \to 0$ as $t \to 0^+$.

2. (a) Solve the initial value problem

$$y' + (\cos t)y^2 = 0, \qquad y(0) = y_0,$$

where y_0 is an arbitrary constant.
(b) For what values of the initial data y_0 is your solution defined for all $-\infty < t < +\infty$?

Solution.

- (a) The equation is separable. We have

$$\begin{aligned} y' &= -(\cos t)y^2, \\ \int \frac{dy}{y^2} &= -\int \cos t\, dt, \\ -\frac{1}{y} &= -\sin t + c. \end{aligned}$$

- Imposing the initial condition, assuming that $y_0 \neq 0$, we get

$$c = -\frac{1}{y_0},$$

- Solving for y, we get

$$y(t) = \frac{1}{1/y_0 + \sin t}$$

  if $y_0 \neq 0$.

- If $y_0 = 0$, then the solution is $y(t) = 0$.
- (b) The solution is defined for all t if $|y_0| < 1$.

3. Consider the ordinary differential equation

$$y' = (y-3)\left(y^2-1\right)$$

(a) Sketch a graph of the right-hand side of this equation as a function of y, and find all equilibrium solutions of the equation.

(b) Sketch the phase line of the equation, and determine the stability of the equilibria you found in (a).

(c) How does the solution with $y(0) = 0$ behave as $t \to +\infty$? How does the solution with $y(0) = 2$ behave as $t \to -\infty$?

Solution.

- (a) The equilibrium points are $y = -1, 1, 3$.
- (b) Trajectories move to the left with increasing t if $y < -1$ or $1 < y < 3$; trajectories move to the right if $-1 < y < 1$ or $y > 3$. The equilibrium $y = 1$ is asymptotically stable. The equilibria $y = -1, 3$ are unstable. (Sketch of phase line omitted.)
- (c) If $y(0) = 0$, then $y(t) \to 1$ as $t \to +\infty$. If $y(0) = 2$, then $y(t) \to 3$ as $t \to -\infty$.

4. (a) Find the general solution of the equation

$$y'' + y' - 2y = 0.$$

(b) Find the general solution of the equation

$$y'' - 2y' + y = 0.$$

Solution.

- (a) The characteristic equation is

  $$r^2 + r - 2 = 0,$$

  with roots $r = -2, 1$.

- The general solution is

  $$y(t) = c_1 e^{-2t} + c_2 e^t.$$

- (b) The characteristic equation is

  $$r^2 - 2r + 1 = 0,$$

  with $r = 1$ as a repeated root.

- The general solution is

  $$y(t) = c_1 e^t + c_2 t e^t.$$

5. Consider the nonhomogeneous ordinary differential equation

$$y'' + 4y = 4t^2 + 10e^{-t}.$$

(a) Find a fundamental pair of solutions for the associated homogeneous equation
(b) Find a particular solution of the nonhomogeneous equation.
(c) Write out the general solution of the nonhomogeneous equation

Solution.

- (a) The characteristic polynomial of the associated homogeneous ODE,

  $$y'' + 4y = 0,$$

  is

  $$r^2 + 4 = 0,$$

  with roots $r = \pm 2i$.

- A fundamental pair of solutions of the homogeneous equation is

  $$y_1(t) = \cos(2t), \qquad y_2(t) = \sin(2t).$$

- (b) Look for a particular solution of the form

  $$y(t) = At^2 + B + Ce^{-t}.$$

  Then

  $$y'' + 4y = 4At^2 + 2A + 4B + 5Ce^{-t}.$$

  We therefore get a solution by choosing $A = 1$, $B = -1/2$, $C = 2$, and

  $$y(t) = t^2 - \frac{1}{2} + 2e^{-t}.$$

- (c) The general solution is

  $$y(t) = c_1 \cos(2t) + c_2 \sin(2t) + t^2 - \frac{1}{2} + 2e^{-t}.$$

6. Consider the ordinary differential equation (in $t > 0$)

$$ty'' - (t+1)y' + y = 0.$$

Note that $y_1(t) = e^t$ is a solution of this equation.
(a) Write $y(t) = v(t)e^t$ and derive a differential equation for $v(t)$.
(b) Solve the equation for $v(t)$ you derived in (a).
(c) Write out an expression for the general solution of the original ordinary differential equation.

Solution.

- (a) Writing $y = ve^t$, we have

$$\begin{aligned} y' &= v'e^t + ve^t, \\ y'' &= v''e^t + 2v'e^t + ve^t, \end{aligned}$$

  and

$$ty'' - (t+1)y' + y = e^t \left\{tv'' + (t-1)\, v'\right\}.$$

- It follows that

$$v'' + \left(1 - \frac{1}{t}\right) v' = 0.$$

- (b) This equation is a first-order linear ODE for v'. The integrating factor is e^t/t and

$$\begin{aligned} &\frac{e^t}{t} v'' + \left(\frac{e^t}{t} - \frac{e^t}{t^2}\right) v' = 0, \\ &\left(\frac{e^t}{t} v'\right)' = 0, \\ &\frac{e^t}{t} v' = c_1, \\ &v' = c_1 t e^{-t}, \\ &v = -c_1 \left(te^{-t} + e^{-t}\right) + c_2. \end{aligned}$$

- The general solution for y is

$$y(t) = c_1(t+1) + c_2 e^t.$$

7. Consider a damped simple harmonic oscillator whose displacement $u(t)$ satisfies the ODE

$$mu'' + \gamma u' + ku = 0,$$

where m, γ, k are positive constants. Let

$$E(t) = \frac{1}{2}m\left(u'\right)^2 + \frac{1}{2}ku^2.$$

Show that if $\gamma > 0$ then

$$E'(t) < 0$$

unless $u'(t) = 0$. Give a physical interpretation of this result.

Solution.

- By the chain rule,

  $$E' = mu'u'' + kuu'.$$

  Using the differential equation to write $mu'' = -\gamma u' - ku$ in this expression, and simplifying the result, we get

  $$\begin{aligned} E' &= -u'\left(\gamma u' + ku\right) + kuu' \\ &= -\gamma\left(u'\right)^2 \\ &< 0, \end{aligned}$$

  unless $u' = 0$. (There was a 'prime' missing in the original question.)

- The quantity E is the total energy of the oscillator (kinetic + potential), and the result shows that damping causes the total energy to decrease in time.

8. Find the general solution (expressed in terms of real-valued functions) of the following 2×2 system

$$\vec{x}'(t) = \begin{pmatrix} -1 & 5 \\ -2 & -3 \end{pmatrix} \vec{x}(t).$$

Solution.

- The characteristic polynomial is

$$\begin{aligned} |A - rI| &= \begin{vmatrix} -1-r & 5 \\ -2 & -3-r \end{vmatrix} \\ &= r^2 + 4r + 13. \end{aligned}$$

  The roots are

$$r = -2 \pm 3i.$$

- Since the system is real, it is sufficient to consider one of these roots, say $r = -2 + 3i$. The corresponding eigenvector is a null-vector of the matrix

$$A - rI = \begin{pmatrix} 1-3i & 5 \\ -2 & -1-3i \end{pmatrix},$$

  which gives

$$\vec{\xi} = \begin{pmatrix} 5 \\ -1+3i \end{pmatrix}.$$

- There are other possible expression for the eigenvector which differ from this one by a constant complex multiple; for example,

$$\vec{\eta} = \begin{pmatrix} 1+3i \\ -2 \end{pmatrix} = \frac{1+3i}{5}\vec{\xi}.$$

  These lead to equivalent, and equally correct, real forms of the solution.

- Writing the corresponding complex exponential solution in real and imaginary parts, we get

$$\vec{x}(t) = e^{(-2+3i)t} \begin{pmatrix} 5 \\ -1+3i \end{pmatrix}$$

$$
\begin{aligned}
&= e^{-2t}\left\{\cos(3t) + i\sin(3t)\right\}\left\{\begin{pmatrix} 5 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right\} \\
&= e^{-2t}\left\{\cos(3t)\begin{pmatrix} 5 \\ -1 \end{pmatrix} - \sin(3t)\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right\} \\
&\quad + ie^{-2t}\left\{\sin(3t)\begin{pmatrix} 5 \\ -1 \end{pmatrix} + \cos(3t)\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right\}.
\end{aligned}
$$

- The general solution is therefore

$$
\begin{aligned}
\vec{x}(t) &= c_1 e^{-2t}\left\{\cos(3t)\begin{pmatrix} 5 \\ -1 \end{pmatrix} - \sin(3t)\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right\} \\
&\quad + c_2 e^{-2t}\left\{\sin(3t)\begin{pmatrix} 5 \\ -1 \end{pmatrix} + \cos(3t)\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right\} \\
&= c_1 e^{-2t}\begin{pmatrix} 5\cos(3t) \\ -\cos(3t) - 3\sin(3t) \end{pmatrix} \\
&\quad + c_2 e^{-2t}\begin{pmatrix} 5\sin(3t) \\ -\sin(3t) + 3\cos(3t) \end{pmatrix}.
\end{aligned}
$$

9. Suppose that

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$. Compute e^{tA}.

Solution.

- We have

$$e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \dots + \frac{1}{k!}t^kA^k + \dots$$

- For the diagonal matrix A, we have

$$A^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix},$$

so

$$e^{tA} = \begin{pmatrix} 1 + t\lambda_1 + \dots + \frac{1}{k!}t^k\lambda_1^k + \dots & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 + t\lambda_n + \dots + \frac{1}{k!}t^k\lambda_n^k + \dots \end{pmatrix}$$

- Since

$$e^{t\lambda} = 1 + t\lambda + \frac{1}{2!}t^2\lambda^2 + \dots + \frac{1}{k!}t^k\lambda^k + \dots,$$

we may write the expression for e^{tA} as

$$e^{tA} = \begin{pmatrix} e^{t\lambda_1} & 0 & \dots & 0 \\ 0 & e^{t\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{t\lambda_n} \end{pmatrix}.$$

10. Suppose that the 2×2 matrix A has the following eigenvalues and eigenvectors:

$$r_1 = -3, \quad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad r_2 = -1, \quad \vec{\xi}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Sketch the trajectories of the system $\vec{x}'(t) = A\vec{x}(t)$, where $\vec{x} = (x_1, x_2)^T$, in the phase plane below.

(b) On the next page, sketch the graphs of $x_1(t)$ and $x_2(t)$ versus t for the solution that satisfies the initial condition $x_1(0) = 3$, $x_2(0) = 1$.

Solution.

- (a) The equilibrium $\vec{x} = 0$ is a stable node. All trajectories approach the origin as $t \to +\infty$. Since $0 > r_2 > r_1$, the trajectories approach the origin tangentially to $\vec{\xi}_2$ (with the exception of those that are exactly in the $\vec{\xi}_1$-direction). As $t \to -\infty$, the trajectories go off to infinity in the direction $\vec{\xi}_1$ (with the exception of those that are exactly in the $\vec{\xi}_2$-direction).

- (b) The solution for $x_1(t)$ decreases monotonically to zero as $t \to +\infty$ from $x_1(0) = 3$. The solution for $x_2(t)$ decreases from $x_2(0) = 1$ and becomes negative, then increases monotonically to 0. (Sketches omitted.)