

ORDINARY DIFFERENTIAL EQUATIONS  
Math 22B-002, Spring 2007  
Final Exam Solutions

1. [20 pts.] (a) Solve the following initial value problem for  $y(t)$ :

$$y' + 2ty = e^{-t^2}, \quad y(0) = y_0.$$

(b) For what initial-value  $y_0$  is  $y(2) = 0$ ?

**Solution.**

- (a) This is a first-order, linear ODE, so we can solve it by the integrating-factor method.
- Multiplication of the ODE by the integrating factor

$$\mu(t) = e^{\int 2t dt} = e^{t^2}$$

gives

$$\left( e^{t^2} y \right)' = 1.$$

Integration of this equation and imposition of the initial condition gives

$$y(t) = (t + y_0) e^{-t^2}.$$

- (b) We have  $y(2) = 0$  if  $y_0 = -2$ .

2. [20 pts.] (a) Solve the initial value problem

$$yy' + 1 = t, \quad y(6) = 3.$$

(b) For what  $t$ -interval is the solution defined?

**Solution.**

- (a) The equation is separable. Separating variables, we get

$$y \, dy = (t - 1) \, dt.$$

- Integrating this equation, and multiplying the result by 2, we get

$$y^2 = (t - 1)^2 + c.$$

- The initial condition implies that  $c = -16$ . After solving for  $y$ , we find that the solution is

$$y(t) = \sqrt{(t - 1)^2 - 16}.$$

- (b) The solution is well-defined and differentiable provided that the quantity inside the square-root is positive, meaning that  $5 < t < \infty$ .

3. [20 pts.] (a) Find the equilibrium solutions of the equation

$$y' = y(y - 2)^3.$$

(b) Sketch the phase line of the equation, and determine the stability of the equilibria you found in (a).

(c) How does the solution with  $y(0) = -1$  behave as  $t \rightarrow +\infty$ ? How does the solution with  $y(0) = 1$  behave as  $t \rightarrow -\infty$ ?

**Solution.**

- (a) The equilibria are  $y = 0$  and  $y = 2$ .
- (b) The function  $f(y) = y(y - 2)^3$  is positive if  $y < 0$  or  $y > 2$ , and negative if  $0 < y < 2$ . Hence, the flow on the phase line is to the right with increasing  $t$  if  $y < 0$  or  $y > 2$ , and to the left if  $0 < y < 2$ . The equilibrium  $y = 0$  is asymptotically stable, and the equilibrium  $y = 2$  is unstable. (A sketch of the phase line is omitted.)
- (c) If  $y(0) = -1$ , then  $y(t) \rightarrow 0$  as  $t \rightarrow +\infty$ . If  $y(0) = 1$ , then  $y(t) \rightarrow 2$  as  $t \rightarrow -\infty$ .

4. [20 pts.] Suppose that  $y_1(t)$ ,  $y_2(t)$ ,  $y_3(t)$  are solutions of the following initial value problems:

$$\begin{aligned} e^t y_1'' + y_1 &= \frac{1}{1+t}, & y_1(0) &= 1, & y_1'(0) &= 0, \\ e^t y_2'' + y_2 &= \frac{1}{1-t}, & y_2(0) &= 0, & y_2'(0) &= 1, \\ e^t y_3'' + y_3 &= \frac{2}{1-t^2}, & y_3(0) &= 1, & y_3'(0) &= 1. \end{aligned}$$

(a) According to general theorems, for what  $t$ -intervals are  $y_1(t)$ ,  $y_2(t)$ ,  $y_3(t)$  uniquely defined?

(b) Express  $y_3(t)$  in terms of  $y_1(t)$ ,  $y_2(t)$ , and justify your answer.

**Solution.**

- (a) Since  $e^t$  is never zero, the continuity of the coefficient functions

$$p(t) = 0, \quad q(t) = e^{-t}, \quad g_1(t) = \frac{e^{-t}}{1+t}, \quad g_2(t) = \frac{e^{-t}}{1-t}, \quad g_3(t) = \frac{2e^{-t}}{1-t^2}$$

implies that  $y_1(t)$  is defined for  $-1 < t < \infty$ ,  $y_2(t)$  is defined for  $-\infty < t < 1$ , and  $y_3(t)$  is defined for  $-1 < t < 1$ .

- Let  $y(t) = y_1(t) + y_2(t)$ . Then, by linearity,

$$\begin{aligned} e^t y'' + y &= e^t y_1'' + y_1 + e^t y_2'' + y_2 \\ &= \frac{1}{1+t} + \frac{1}{1-t} = \frac{2}{1-t^2}, \end{aligned}$$

and

$$\begin{aligned} y(0) &= y_1(0) + y_2(0) = 1, \\ y'(0) &= y_1'(0) + y_2'(0) = 1. \end{aligned}$$

It follows that  $y(t)$  satisfies the same IVP as  $y_3(t)$ , so  $y(t) = y_3(t)$  by the existence-uniqueness theorem, and

$$y_3(t) = y_1(t) + y_2(t) \quad \text{for } -1 < t < 1.$$

5. [20 pts.] (a) Find the general solution of the equation

$$y'' - 4y' + 3y = 0.$$

(b) Find the general solution of the equation

$$2y'' + 2y' + 5y = 0.$$

**Solution.**

- (a) The characteristic equation is

$$r^2 - 4r + 3 = 0,$$

with solutions

$$r = 1, 3.$$

The general solution is

$$y(t) = c_1 e^t + c_2 e^{3t},$$

where  $c_1, c_2$  are arbitrary constants.

- (b) The characteristic equation is

$$2r^2 + 2r + 5 = 0,$$

with solutions

$$r = -\frac{1}{2} \pm i\frac{3}{2}.$$

The general solution is

$$y(t) = c_1 e^{-t/2} \cos\left(\frac{3t}{2}\right) + c_2 e^{-t/2} \sin\left(\frac{3t}{2}\right),$$

where  $c_1, c_2$  are arbitrary constants.

6. [20 pts.] Suppose that  $\omega_0$ ,  $\omega$ ,  $F_0$  are nonzero constants, and consider the ordinary differential equation

$$y'' + \omega_0^2 y = F_0 \cos(\omega t).$$

- (a) Find a particular solution for  $y(t)$  if  $\omega \neq \omega_0$ .
- (b) Find a particular solution for  $y(t)$  if  $\omega = \omega_0$ .
- (c) Give a brief physical interpretation of these solutions for oscillators.

**Solution.**

- (a) If  $\omega \neq \omega_0$  (and also assuming  $\omega \neq -\omega_0$ ), then, using the method of undetermined coefficients, we look for a solution of the form

$$y(t) = A \cos \omega t.$$

In that case,

$$y'' + \omega_0^2 y = (-\omega^2 + \omega_0^2) A \cos \omega t.$$

We obtain a particular solution if  $(-\omega^2 + \omega_0^2) A = F_0$ , meaning that

$$y(t) = \frac{F_0}{\omega_0^2 - \omega^2} \cos \omega t.$$

(b) If  $\omega = \omega_0$ , then a solution of the form assumed in (a) does not work. Instead, we look for a particular solution of the form

$$y(t) = At \sin \omega_0 t.$$

Then we compute that

$$y'' + \omega_0^2 y = 2A\omega_0 \cos \omega_0 t.$$

We obtain a particular solution if  $2A\omega_0 = F_0$ , meaning that

$$y(t) = \frac{F_0}{2\omega_0} t \sin \omega_0 t.$$

(c) These solutions illustrate the phenomenon of resonance. If an undamped simple harmonic oscillator is forced sinusoidally by an external force whose frequency  $\omega$  is different from the natural frequency  $\omega_0$  of the oscillator, then the response of the oscillator is also sinusoidal, with large amplitude if  $\omega$  is close to  $\omega_0$ . If the frequency of the external force is equal to the natural frequency of the oscillator ( $\omega = \omega_0$ ), then the amplitude of the oscillator grows linearly in time.

7. [20 pts.] Find the general solution of the following  $2 \times 2$  system, and express your answer in terms of real-valued functions:

$$\vec{x}'(t) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}(t).$$

**Solution.**

- The characteristic polynomial of the coefficient matrix is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & 1 \\ -1 & -\lambda \end{vmatrix} \\ &= \lambda^2 - \lambda + 1 \end{aligned}$$

Solving the equation  $\lambda^2 - \lambda + 1 = 0$ , we find that the eigenvalues are

$$\lambda = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

- For a system with real coefficients, it is sufficient to consider only one of the eigenvalues in a complex conjugate pair, say  $\lambda = 1/2 + i\sqrt{3}/2$ . Then

$$A - \lambda I = \begin{pmatrix} 1/2 - i\sqrt{3}/2 & 1 \\ -1 & -1/2 - i\sqrt{3}/2 \end{pmatrix}.$$

An eigenvector is

$$\vec{\xi} = \begin{pmatrix} 1 \\ -1/2 + i\sqrt{3}/2 \end{pmatrix}.$$

- The corresponding solution is

$$\begin{aligned} e^{\lambda t} \vec{\xi} &= e^{(1/2+i\sqrt{3}/2)t} \begin{pmatrix} 1 \\ -1/2 + i\sqrt{3}/2 \end{pmatrix} \\ &= e^{t/2} \left[ \cos\left(\frac{\sqrt{3}}{2}t\right) + i \sin\left(\frac{\sqrt{3}}{2}t\right) \right] \left[ \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + i \begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix} \right] \\ &= e^{t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} - e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) \begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix} \\ &\quad + i \left\{ e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + e^{t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) \begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix} \right\} \end{aligned}$$

- The real and imaginary parts of this complex solution form a fundamental pair of real-valued solutions for the ODE. The general solution is

$$\begin{aligned}\vec{x}(t) = & c_1 e^{t/2} \left\{ \cos\left(\frac{\sqrt{3}}{2}t\right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} - \sin\left(\frac{\sqrt{3}}{2}t\right) \begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix} \right\} \\ & + c_2 e^{t/2} \left\{ \sin\left(\frac{\sqrt{3}}{2}t\right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + \cos\left(\frac{\sqrt{3}}{2}t\right) \begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix} \right\},\end{aligned}$$

where  $c_1$  and  $c_2$  are arbitrary constants.

8. [20 pts.] Suppose that a  $2 \times 2$  matrix  $A$  has the following eigenvalues and eigenvectors:

$$r_1 = -2, \quad \vec{\xi}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}; \quad r_2 = 1, \quad \vec{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(a) Classify the equilibrium  $\vec{x} = 0$  (node, saddle, spiral, center). Is it stable or unstable?

(b) Sketch the trajectories of the system  $\vec{x}' = A\vec{x}$ , where  $\vec{x} = (x_1, x_2)^T$ , in the phase plane below.

(c) On the next page, sketch the graphs of  $x_1(t)$  and  $x_2(t)$  versus  $t$  for the solution that satisfies the initial condition  $x_1(0) = 1, x_2(0) = 1$ .

**Solution.**

- (a) The equilibrium is a saddle point (two real eigenvalues with opposite signs), which is unstable.
- Sketches for (b), (c) are omitted.

9. [20 pts.] Recall that

$$e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \dots + \frac{1}{k!}t^kA^k + \dots$$

(a) Compute  $e^{tA}$  explicitly if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

HINT. Recall the expansions:  $e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \dots + \frac{1}{k!}t^k + \dots$ ;  $e^{-t} = 1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \dots + \frac{(-1)^k}{k!}t^k + \dots$

(b) Use your result from (a) to write out the solution  $\vec{x}(t) = (x_1(t), x_2(t))^T$  of the initial value problem:

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

For what initial values  $(c_1, c_2)^T$  is it true that  $\vec{x}(t) \rightarrow 0$  as  $t \rightarrow +\infty$ ?

**Solution.**

- (a) We compute that

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

It follows that  $A^3 = A^2 \cdot A = I \cdot A = A$ ,  $A^4 = A^2 \cdot A^2 = I \cdot I = I$ , and so on. In general,

$$A^k = I \quad \text{if } k \text{ is even,} \quad A^k = A \quad \text{if } k \text{ is odd.}$$

- The series for  $e^{tA}$  therefore becomes

$$e^{tA} = \left(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \dots\right) I + \left(t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \dots\right) A.$$

Adding and subtracting the power series expansions of  $e^t$  and  $e^{-t}$ , we find that

$$\begin{aligned} \frac{1}{2}(e^t + e^{-t}) &= 1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \dots, \\ \frac{1}{2}(e^t - e^{-t}) &= t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \dots \end{aligned}$$

It follows that

$$\begin{aligned}
 e^{tA} &= \frac{1}{2}(e^t + e^{-t})I + \frac{1}{2}(e^t - e^{-t})A \\
 &= \frac{1}{2}(e^t + e^{-t}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(e^t - e^{-t}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
 &= \frac{1}{2}e^t \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}e^{-t} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
 \end{aligned}$$

- (b) The solution is  $\vec{x}(t) = e^{tA}\vec{x}(0)$ , which gives

$$\begin{aligned}
 \vec{x}(t) &= \frac{1}{2}e^t \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \frac{1}{2}e^{-t} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
 &= \frac{1}{2}e^t (c_1 + c_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-t} (c_1 - c_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
 \end{aligned}$$

- The solution approaches 0 as  $t \rightarrow +\infty$  if the coefficient of term proportional to  $e^t$  is 0. This is the case when

$$c_1 + c_2 = 0,$$

meaning that

$$\vec{x}(0) = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

for some scalar  $c$ .

**Remark.** We can write these results in more a compact form by use of the hyperbolic cosine and sine,

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}.$$

Then

$$\begin{aligned}
 e^{tA} &= \cosh t I + \sinh t A \\
 &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},
 \end{aligned}$$

and the solution of the ODE is

$$\vec{x}(t) = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}.$$

Note the analogy between these expressions for the exponential of the symmetric matrix  $A$  with real eigenvalues, and the exponential of the anti-symmetric matrix

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with imaginary eigenvalues that we considered in class. We showed there that

$$\begin{aligned} e^{tB} &= \cos t I + \sin t B \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \end{aligned}$$

10. [20 pts.] (a) Use the method of variation of parameters to solve the ODE

$$y'' - y = g(t),$$

where  $g(t)$  is a given continuous function. That is, write

$$y(t) = e^t u_1(t) + e^{-t} u_2(t), \quad e^t u_1'(t) + e^{-t} u_2'(t) = 0.$$

Solve for  $u_1'(t)$ ,  $u_2'(t)$ , and find the general solution for  $y(t)$ .

(b) Show that you can write the solution of the initial value problem

$$y'' - y = g(t), \quad y(0) = 0, \quad y'(0) = 0$$

as a convolution

$$y(t) = \int_0^t K(t-s)g(s) ds,$$

and determine the function  $K(t)$ .

**Solution.**

- (a) Using the product rule and the equation satisfied by  $u_1'$ ,  $u_2'$ , we compute that

$$\begin{aligned} y' &= e^t u_1' + e^{-t} u_2' + e^t u_1 - e^{-t} u_2 \\ &= e^t u_1 - e^{-t} u_2. \end{aligned}$$

Differentiating this equation, we get

$$y'' = e^t u_1' - e^{-t} u_2' + e^t u_1 + e^{-t} u_2.$$

It follows that

$$y'' - y = e^t u_1' - e^{-t} u_2'.$$

Thus,  $y$  satisfies the ODE if  $u_1$ ,  $u_2$  satisfy

$$\begin{aligned} e^t u_1' - e^{-t} u_2' &= g(t), \\ e^t u_1' + e^{-t} u_2' &= 0. \end{aligned}$$

- Solving this pair of linear equations for  $u'_1$ ,  $u'_2$ , we get

$$\begin{aligned}u'_1(t) &= \frac{1}{2}e^{-t}g(t), \\u'_2(t) &= -\frac{1}{2}e^t g(t).\end{aligned}$$

Integrating these equations, we get

$$\begin{aligned}u_1(t) &= \frac{1}{2} \int_0^t e^{-s}g(s) ds + c_1, \\u_2(t) &= -\frac{1}{2} \int_0^t e^s g(s) ds + c_2,\end{aligned}$$

where  $c_1$ ,  $c_2$  are constants of integration.

- Using these expressions in the equation  $y = e^t u_1 + e^{-t} u_2$ , we find that the general solution for  $y$  is

$$y(t) = \frac{1}{2}e^t \int_0^t e^{-s}g(s) ds - \frac{1}{2}e^{-t} \int_0^t e^s g(s) ds + c_1 e^t + c_2 e^{-t}.$$

- (b) From the expressions

$$y = e^t u_1 + e^{-t} u_2, \quad y' = e^t u_1 - e^{-t} u_2,$$

we see that  $y(0) = y'(0) = 0$  if and only if  $u_1(0) = u_2(0) = 0$  which implies that  $c_1 = c_2 = 0$ . The solution of the initial value problem for  $y(t)$  is then

$$\begin{aligned}y(t) &= \frac{1}{2}e^t \int_0^t e^{-s}g(s) ds - \frac{1}{2}e^{-t} \int_0^t e^s g(s) ds \\&= \frac{1}{2} \int_0^t e^{t-s}g(s) ds - \frac{1}{2} \int_0^t e^{-t+s}g(s) ds \\&= \int_0^t \left[ \frac{e^{t-s} - e^{-(t-s)}}{2} \right] g(s) ds \\&= \int_0^t \sinh(t-s)g(s) ds\end{aligned}$$

- This solution has the required convolution form, with

$$\begin{aligned} K(t) &= \frac{e^t - e^{-t}}{2} \\ &= \sinh t. \end{aligned}$$

**Remark.** Note the analogy with the problem

$$\begin{aligned} y'' + y &= g(t), \\ y(0) = y'(0) &= 0, \end{aligned}$$

in which the homogeneous ODE has  $\cos t$ ,  $\sin t$  as solutions instead of  $e^t$ ,  $e^{-t}$ . As we showed in class, by the use of variation of parameters, the solution is

$$y(t) = \int_0^t \sin(t-s)g(s) ds.$$