Midterm 1: Solutions
Math 22B, Spring 2007

1. [10%] Say if the following ODEs are linear or nonlinear. By use of the theorems given in class, what can you say about the existence and uniqueness of a solution $y(t)$ of the ODEs with the initial condition $y(1) = -2$?
(a) $(t+1)y' + (\cos t)y = e^t$.
(b) $(y+1)y' + (\cos t)y = e^t$.

Solution.

- (a) Linear $[y' + p(t)y = g(t)]$. The coefficient functions

  $$p(t) = \frac{\cos t}{t+1}, \qquad g(t) = \frac{e^t}{t+1}$$

  are continuous in the interval $-1 < t < \infty$, containing the initial time $t_0 = 1$, so a unique solution exists in $-1 < t < \infty$.

- (b) Nonlinear $[y' = f(t,y)]$. The function

  $$f(t,y) = \frac{-(\cos t)y + e^t}{y+1}$$

  is continuous and has a continuous derivative with respect to y except when $y = -1$. Since the initial value of y is -2, a unique solution exists in some time interval containing $t_0 = 1$.

2. [10%] Suppose that for certain continuous function $p(t)$, $g(t)$ and initial time t_0, the functions $y_1(t)$, $y_2(t)$ are solutions of the initial value problems

$$\begin{aligned} &y_1' + p(t)y_1 = g(t), \qquad && y_1(t_0) = 0, \\ &y_2' + p(t)y_2 = 0, && y_2(t_0) = 1. \end{aligned}$$

If c_1, c_2 are constants, show that the function

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is the solution of the initial value problem

$$y' + p(t)y = c_1 g(t), \qquad y(t_0) = c_2.$$

Solution.

- Using the definition of $y(t)$, the linearity of the derivative, and the ODEs satisfied by $y_1(t)$, $y_2(t)$, we get

$$\begin{aligned} y' + p(t)y &= (c_1y_1 + c_2y_2)' + p(t)\,(c_1y_1 + c_2y_2) \\ &= c_1y_1' + c_2y_2' + c_1p(t)y_1 + c_2p(t)y_2 \\ &= c_1\,(y_1' + p(t)y_1) + c_2\,(y_2' + p(t)y_2) \\ &= c_1 \cdot g(t) + c_2 \cdot 0 \\ &= c_1 g(t). \end{aligned}$$

- Similarly, using the definition of $y(t)$ and the initial conditions satisfied by $y_1(t)$, $y_2(t)$, we get

$$\begin{aligned} y(t_0) &= c_1y_1(t_0) + c_2y_2(t_0) \\ &= c_1 \cdot 0 + c_2 \cdot 1 \\ &= c_2. \end{aligned}$$

3. [20%] Suppose that a, b are constants. Find the general solution of

$$y' + ay = b.$$

If $a > 0$, how does the solution behave as $t \to +\infty$?

Solution.

- Multiplying the ODE by the integrating factor $\mu(t) = e^{at}$ and rewriting the result, we get
$$\left(e^{at}y\right)' = be^{at}.$$
Integrating this equation and solving for $y(t)$, we get
$$y(t) = \frac{b}{a} + ce^{-at} \qquad \text{if } a \neq 0,$$
and
$$y(t) = bt + c \qquad \text{if } a = 0,$$
where c is a constant of integration.

- If $a > 0$, then $e^{-at} \to 0$ as $t \to +\infty$, so
$$y(t) \to \frac{b}{a} \qquad \text{as } t \to +\infty.$$

Remark. This equation is the general form of an autonomous *linear* first-order scalar ODE. If $a \neq 0$, it has a single equilibrium, $y = b/a$. This equilibrium is asymptotically stable if $a > 0$ and unstable if $a < 0$. Note that, unlike autonomous nonlinear ODEs, an autonomous linear ODE cannot have finitely many different equilibria. (If $a = 0$ and $b \neq 0$, the ODE has no equilibria, and if $a = b = 0$ every point $y = c$ is an equilibrium.)

4. [20%] (a) Find the solution of the initial value problem

$$\begin{aligned} &ty' + 3y = \frac{1}{t}, \qquad t > 0, \\ &y(1) = y_0. \end{aligned}$$

(b) For what initial values y_0 is the solution $y(t)$ equal to zero for some $t > 0$?

Solution.

- (a) The ODE is linear and first-order, so we can solve it by the integrating factor method. The standard form is
$$y' + \frac{3}{t}y = \frac{1}{t^2}.$$
Multiplying the standard form of the ODE by the integrating factor
$$\mu(t) = \exp\left(\int \frac{3}{t}\,dt\right) = \exp\left(3\ln t\right) = t^3$$
and rewriting the left-hand side of the result as an exact derivative, we get
$$\left(t^3 y\right)' = t.$$
Integrating this equation and solving for y, we get
$$y(t) = \frac{1}{2t} + \frac{C}{t^3},$$
where C is a constant of integration.

- Imposing the initial condition $y = y_0$ when $t = 1$, and solving for C, we find that
$$C = y_0 - \frac{1}{2}.$$
Thus, the solution is
$$y(t) = \frac{1}{2t} + \frac{y_0 - 1/2}{t^3}.$$

- (b) We have
$$y(t) = \frac{1}{t}\left(\frac{1}{2} + \frac{y_0 - 1/2}{t^2}\right).$$
This vanishes at some $t > 0$ if and only if $y_0 - 1/2 < 0$ or
$$y_0 < \frac{1}{2}.$$

5. [20%] (a) Solve the initial value problem

$$\begin{aligned} &ty' + y^2 = 0, \\ &y(1) = 1. \end{aligned}$$

(b) What is the largest t-interval in which the solution exists?

Solution.

- (a) The equation is nonlinear and separable. Separating variables, we get
$$-\int \frac{dy}{y^2} = \int \frac{dt}{t}.$$
Integration of this equation gives
$$\frac{1}{y} = \ln t + C.$$
(Here, we write the integral $\ln|t|$ as $\ln t$, since the initial time $t_0 = 1$ is positive.)

- The initial condition, $y = 1$ at $t = 1$, implies that $C = 1$. Solving for y, we find that the solution is
$$y(t) = \frac{1}{\ln t + 1}.$$

- (b) The denominator vanishes when $\ln t = -1$ or $t = 1/e > 0$. The solution is therefore defined in the time interval
$$\frac{1}{e} < t < +\infty.$$

6. [20%] Consider the ODE

$$y' = y^2 - y^4.$$

(a) Find all equilibrium solutions.

(b) Sketch the phase line.

(c) Determine the stability of the equilibria you found in (a).

Solution.

- (a) The equilibria satisfy

  $$y^2 - y^4 = y^2\left(1 - y^2\right) = 0,$$

  so $y = 0$ or $y = \pm 1$.

- (b) The phase line looks like this:

  $$\begin{array}{c} \longleftarrow \cdot \longrightarrow \quad \cdot \quad \longrightarrow \cdot \longleftarrow \\ 0 \end{array}$$

- (c) $y = 0$ is semistable, $y = 1$ is asymptotically stable, and $y = -1$ is unstable.