ORDINARY DIFFERENTIAL EQUATIONS
Math 22B-002, Spring 2008
Midterm 1: Solutions

NAME..

SIGNATURE..

I.D. NUMBER..

No books, notes, or calculators.
Unless stated otherwise, show all your work.

Question	Points	Score
1	10	
2	10	
3	20	
4	20	
5	20	
6	20	
Total	100	

1. [10%] By use of the general theorems given in class, say what you can about the existence, uniqueness, and t-interval of definition of a solution $y(t)$ of the ODE

$$(\ln|t|)\, y' + e^t y = \frac{t^2 - 9}{t - 4}$$

with the initial conditions: (a) $y(2) = 5$; (b) $y(5) = 2$. (Don't try to solve the ODE explicitly.)

SOLUTION

- The equation is linear, of the form

  $$y' + p(t)y = g(t),$$

  with coefficient functions

  $$p(t) = \frac{e^t}{\ln|t|}, \qquad g(t) = \frac{t^2 - 9}{(t-4)\ln|t|}.$$

  The function $p(t)$ is discontinuous at $t = 0$, where $\ln|t|$ is discontinuous and at $t = \pm 1$, where $\ln|t| = 0$. The function $g(t)$ is discontinuous at $t = 0, \pm 1$ and at $t = 4$, where $t - 4 = 0$.

- (a) By the general existence-uniqueness theorem for linear IVPs, a unique solution $y(t)$ exists and is defined in the interval $1 < t < 4$, which is the largest t-interval containing the initial time $t_0 = 2$ where the coefficient functions are continuous.

- (b) Similarly, a unique solution $y(t)$ exists and is defined in the interval $4 < t < \infty$, which is the largest t-interval containing $t_0 = 5$ where the coefficient functions are continuous.

2. [10%] Suppose that for certain continuous functions $p(t)$, $g(t)$ the functions $y_1(t)$, $y_2(t)$ are solutions of the ODEs

$$y_1' + p(t)y_1 = 0, \qquad y_2' + p(t)y_2 = g(t).$$

Give an expression for the general solution $y(t)$ of the ODE

$$y' + p(t)y = 3g(t)$$

in terms of $y_1(t)$ and $y_2(t)$, and justify your answer.

SOLUTION

- The general solution is

  $$y(t) = Cy_1(t) + 3y_2(t),$$

  where C is an arbitrary constant.

- To show that $y(t)$ is a solution of the ODE, we compute

  $$\begin{aligned} y' + py &= (Cy_1 + 3y_2)' + p\,(Cy_1 + 3y_2) \\ &= Cy_1' + 3y_2' + Cpy_1 + 3py_2 \\ &= C\,(y_1' + py_1) + 3\,(y_2' + py_2) \\ &= C \cdot 0 + 3 \cdot g \\ &= 3g. \end{aligned}$$

- The solution contains an arbitrary constant of integration, so it is the general solution.

REMARK. We should have included the stipulation that $y_1(t)$ is a *nonzero* solution of the homogeneous equation in order to get the general solution. In that case, if $y_1(t_0) \neq 0$, we get the solution $y(t)$ with $y(t_0) = y_0$ by choosing

$$C = \frac{y(t_0) - 3y_2(t_0)}{y_1(t_0)}.$$

3. [20%] (a) Find the solution of the initial value problem

$$\begin{aligned} y' &= y \ln y, \\ y(0) &= e. \end{aligned}$$

HINT. Use the substitution $u = \ln y$.
(b) For what values of t is the solution defined?

SOLUTION

- Separating variables we get

  $$\int \frac{dy}{y \ln y} = \int dt.$$

  The substitution $u = \ln y$ gives $du = dy/y$ and

  $$\int \frac{dy}{y \ln y} = \int \frac{du}{u} = \ln u + C = \ln(\ln y) + C.$$

  Hence,

  $$\ln(\ln y) = t + C.$$

- At $t = 0$, we have $y = e$, so

  $$C = \ln(\ln e) = \ln 1 = 0.$$

- Solving for y, we get

  $$y(t) = e^{e^t}.$$

- (b) The solution is defined for all $-\infty < t < \infty$.

REMARK. This ODE is interesting because, although the right-hand side grows faster than a linear function of y as $y \to \infty$, the solution exists for all t. It grows very quickly (doubly-exponentially) as $t \to \infty$. This ODE has close to the maximal growth in y that allows solutions to be defined for all t. For example, a similar calculation shows that any solution of

$$y' = y(\ln y)^n$$

with initial data $y(0) = y_0 > 1$ goes to infinity in finite time for any $n > 1$.

4. [20%] (a) Solve the initial value problem

$$\begin{aligned} y' + y &= t, \\ y(0) &= 1. \end{aligned}$$

(b) Find the time t at which the solution attains its minimum value.

SOLUTION

- (a) This is a linear, nonhomogeneous first-order ODE. The integrating factor is
$$\mu(t) = e^{\int 1\,dt} = e^t$$
Multiplying the equation by e^t and rearranging the left-hand side, we get
$$\left(e^t y\right)' = te^t.$$
Integrating this equation, and using an integration by parts, we find that
$$\begin{aligned} e^t y &= \int te^t\,dt \\ &= te^t - \int e^t\,dt \\ &= te^t - e^t + C. \end{aligned}$$

- Imposing the initial condition, we get
$$1 = -1 + C,$$
so $C = 2$.

- Solving for $y(t)$, we obtain that
$$y(t) = t - 1 + 2e^{-t}.$$

- (b) Differentiating the solution, we get
$$y'(t) = 1 - 2e^{-t}, \qquad y''(t) = 2e^t.$$
If follows that $y'(t) = 0$ if $e^t = 2$ or
$$t = \ln 2.$$
This critical point corresponds to a minimum of $y(t)$ since $y''(t) > 0$.

5. [20%] (a) Solve the initial value problem

$$\begin{aligned} y' &= e^{-y}\cos t, \\ y(0) &= y_0. \end{aligned}$$

(b) For what initial values y_0 is the solution $y(t)$ defined for all $-\infty < t < \infty$?

SOLUTION

- (a) The equation is nonlinear and separable. Separating variables we get
$$\int e^y\,dy = \int \cos t\,dt.$$
Evaluation of the integrals gives
$$e^y = \sin t + C.$$
Imposing the initial condition (this can be done now or after solving for y), we get
$$C = e^{y_0}.$$
The solution for y is therefore
$$y(t) = \ln\left(\sin t + e^{y_0}\right).$$

- (b) Since $\ln x$ is differentiable if $x > 0$ and $\sin t \geq -1$, the solution is defined for all $-\infty < t < \infty$ if and only if $e^{y_0} > 1$, meaning that
$$y_0 > 0.$$

6. [20%] (a) Find all equilibria of the ODE

$$y' = \frac{y^2 - 1}{y^2 + 1}.$$

(b) Sketch the phase line, and determine the stability of the equilibria you found in (a).

(c) Sketch the graph versus t of the solution $y(t)$ that satisfies the initial condition $y(0) = 0$.

SOLUTION

- (a) The equilibria satisfy $f(y) = 0$, where

  $$f(y) = \frac{y^2 - 1}{y^2 + 1},$$

  which implies that $y^2 = 1$. Therefore the equilibria are

  $$y = -1, \qquad y = 1.$$

- (b) The denominator $y^2 + 1$ is always positive, so the function $f(y)$ is positive for $y > 1$ or $y < -1$, and negative for $-1 < y < 1$. The phase line therefore looks like this:

  $$\longrightarrow \cdot \longleftarrow \cdot \longrightarrow$$

  The equilibrium $y = -1$ is asymptotically stable, and the equilibrium $y = 1$ is unstable.

- (c) The graph of $y(t)$ has negative slope everywhere. It asymptotes to the line $y = -1$ as $t \to +\infty$ and the line $y = 1$ as $t \to -\infty$. The sketch is omitted.