ORDINARY DIFFERENTIAL EQUATIONS

Math 22B-003, Spring 2006

Midterm 2

NAME...

SIGNATURE...

I.D. NUMBER..

No books, notes, or calculators.
Read the questions carefully.
Show all your work.

Question	Points	Score
1	10	
2	10	
3	20	
4	20	
5	20	
6	20	
Total	100	

1. [10%] Find the general solution of the ODE

$$y'' - 6y' + 13y = 0.$$

Solution.

- Characteristic equation

$$r^2 - 6r + 13 = 0.$$

- Solution (by quadratic formula or completing the square)

$$r = 3 \pm 2i.$$

- General solution of the ODE

$$y(t) = c_1 e^{3t} \cos 2t + c_2 e^{3t} \sin 2t,$$

where c_1, c_2 are arbitrary constants.

2. [10%] Suppose that $\{y_1, y_2, y_3\}$ are solutions of the ODE

$$y'' + e^t y' + t^2 y = 0$$

that satisfy the following initial conditions:

$$\begin{aligned} y_1(0) &= 1, & y_1'(0) &= -1; \\ y_2(0) &= -1, & y_2'(0) &= 1; \\ y_3(0) &= -1, & y_3'(0) &= -1; \end{aligned}$$

For what values of t are $y_i(t)$, $i = 1, 2, 3$, defined? Which of the following sets form a fundamental set of solutions for the ODE: (a) $\{y_1, y_2\}$; (b) $\{y_2, y_3\}$; (c) $\{y_1, y_3\}$? Explain your answers.

Solution.

- By the existence theorem for linear ODEs, the solutions are defined for all values of $-\infty < t < \infty$, since the coefficient functions e^t and t^2 are continuous for all t.

- Two solutions $\{y_i, y_j\}$ form a fundamental solution set if they are linearly independent, which is the case if their Wronskian

$$W(y_i, y_j)(t) = \begin{vmatrix} y_i(t) & y_j(t) \\ y_i'(t) & y_j'(t) \end{vmatrix}$$

is nonzero at $t = 0$.

- (a) *Not* a fundamental set of solutions, since

$$W(y_1, y_2)(0) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0.$$

Or you can observe that y_1 and $-y_2$ satisfy the same initial value problem, so the uniqueness theorem for initial value problems implies that $y_1(t) = -y_2(t)$ for all t, meaning that y_1, y_2 are linearly dependent.

- (b) Fundamental set of solutions, since

$$W(y_2, y_3)(0) = \begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix} = 2.$$

- (c) Fundamental set of solutions, since

$$W(y_1, y_3)(0) = \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -2.$$

3. [20%] (a) Find the solution $y(t)$ of the initial value problem

$$\begin{aligned} &y'' - y' - 2y = 0, \\ &y(0) = 3, \qquad y'(0) = a, \end{aligned}$$

where a is an arbitrary constant.
(b) For what value of a does $y(t) \to 0$ as $t \to +\infty$?

Solution.

- (a) Characteristic equation

$$\begin{aligned} &r^2 - r - 2 = 0 \\ &(r+1)(r-2) = 0 \\ &r = -1, 2. \end{aligned}$$

  General solution

$$y(t) = c_1 e^{-t} + c_2 e^{2t}.$$

- Initial conditions imply

$$\begin{aligned} c_1 + c_2 &= 3, \\ -c_1 + 2c_2 &= a. \end{aligned}$$

  Solution for constants

$$c_1 = 2 - \frac{a}{3}, \qquad c_2 = 1 + \frac{a}{3},$$

  so the solution of the initial value problem is

$$y(t) = \left(2 - \frac{a}{3}\right) e^{-t} + \left(1 + \frac{a}{3}\right) e^{2t}.$$

- (b) Solution decays to zero as $t \to +\infty$ if coefficient c_2 of e^{2t} vanishes, or

$$a = -3.$$

4. [20%] (a) Find a particular solution of the ODE

$$y'' + 4y' + 4y = 4t^2.$$

(b) Find the general solution of the ODE.

Solution.

- (a) Look for particular solution Y of form

  $$Y(t) = at^2 + bt + c,$$

  where a, b, c are constants. Then

  $$Y'(t) = 2at + b, \qquad Y''(t) = 2a$$

  and

  $$Y'' + 4Y' + 4Y = 4at^2 + (8a + 4b)t + (2a + 4b + 4c).$$

- We get a solution if

  $$4a = 4, \quad 8a + 4b = 0, \quad 2a + 4b + 4c = 0,$$

  which implies that

  $$a = 1, \quad b = -2, \quad c = \frac{3}{2}.$$

  This gives the particular solution

  $$Y(t) = t^2 - 2t + \frac{3}{2}.$$

- (b) Characteristic equation of associated homogeneous ODE is

  $$\begin{aligned} &r^2 + 4r + 4 = 0, \\ &(r + 2)^2 = 0, \\ &r = -2 \qquad \text{(repeated)}. \end{aligned}$$

  Therefore a fundamental set of solutions for the homogenous ODE is

  $$\left\{e^{-2t}, te^{-2t}\right\}.$$

- By the superposition principle, the general solution of the nohomogeneous ODE is

  $$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + t^2 - 2t + \frac{3}{2}$$

  where c_1, c_2 are arbitrary constants.

5. [20%] (a) Verify that $y(t) = t^2$ is a solution of the ODE

$$t^2 y'' - 2y = 0 \qquad (t > 0).$$

(b) Use the method of reduction of order to find a fundamental set of solutions for the ODE.

Solution.

- (a) If $y(t) = t^2$, then

$$t^2 y'' - 2y = t^2(2) - 2(t^2) = 0,$$

  so it's a solution.

- (b) Write

$$y(t) = t^2 v(t).$$

  Then

$$\begin{aligned} y' &= t^2 v' + 2tv, \\ y'' &= t^2 v'' + 4tv' + 2v, \\ t^2 y'' - 2y &= t^2 \left(t^2 v'' + 4tv' + 2v\right) - 2t^2 v \\ &= t^4 v'' + 4t^3 v'. \end{aligned}$$

  Therefore

$$t^4 v'' + 4t^3 v' = 0.$$

- Either notice that the left-hand side of this equation is an exact derivative, so

$$\left(t^4 v'\right)' = 0,$$

  and integrate, or rewrite the equation as first-order ODE in standard form for $w = v'$,

$$w' + \frac{4}{t} w = 0.$$

  The integrating factor is

$$\mu(t) = \exp \int \frac{4}{t}\, dt = \exp(4 \ln t) = t^4$$

Hence

$$\begin{aligned}&t^4 w' + 4t^3 w = 0\\&\left(t^4 w\right)' = 0\\&t^4 w = c_1.\end{aligned}$$

- Either way, we find that
$$v' = \frac{c_1}{t^4}.$$
Integrating this equation, and redefining the arbitrary constant c_1, we get
$$v = \frac{c_1}{t^3} + c_2.$$

- It follows that $y = t^2 v$ is given by
$$y(t) = \frac{c_1}{t} + c_2 t^2,$$
so
$$y_1(t) = \frac{1}{t}, \qquad y_2(t) = t^2$$
are a pair of solutions.

- These solutions are linearly independent in $t > 0$, since their Wronskian
$$W(t) = \begin{vmatrix} 1/t & t^2 \\ -1/t^2 & 2t \end{vmatrix} = 3$$
is nonzero, so a fundamental set of solutions of the ODE is
$$\left\{\frac{1}{t}, t^2\right\}.$$

6. [20%] Suppose that ω is a positive constant and $\omega \neq 1$.

(a) Solve the initial value problem

$$y'' + y = \sin \omega t,$$
$$y(0) = 0, \qquad y'(0) = 0.$$

(b) Using L'Hôspital's rule, or otherwise, find the limit of the solution as $\omega \to 1$. Comment briefly on the result.

Solution.

- (a) Look for a particular solution Y of the nonhomogeneous ODE of the form
$$Y(t) = a \sin \omega t.$$
Then
$$Y'' + Y = a(1-\omega^2) \sin \omega t$$
so we want
$$a(1-\omega^2) = 1.$$
Therefore $a = 1/(1-\omega^2)$ and a particular solution is
$$Y(t) = \frac{1}{1-\omega^2} \sin \omega t.$$
(The denominator is nonzero since $\omega > 0$ and $\omega \neq 1$.)

- The characteristic equation of the associated homogeneous equation $y'' + y = 0$ is
$$r^2 + 1 = 0$$
with roots $r = \pm i$. The corresponding homogenous solutions are $y_1(t) = \cos t$, $y_2(t) = \sin t$.

- By the superposition principle, the general solution of the nonhomogeneous ODE is
$$y(t) = c_1 \cos t + c_2 \sin t + \frac{1}{1-\omega^2} \sin \omega t$$

- Imposing the initial conditions on the general solution, we get

$$\begin{aligned} c_1 &= 0, \\ c_2 + \frac{\omega}{1-\omega^2} &= 0 \end{aligned}$$

It follows that $c_1 = 0$ and $c_2 = -\omega/(1-\omega^2)$, so the solution of the initial value problem is

$$\begin{aligned} y(t) &= -\frac{\omega}{1-\omega^2}\sin t + \frac{1}{1-\omega^2}\sin \omega t \\ &= \frac{\sin \omega t - \omega \sin t}{1-\omega^2}. \end{aligned}$$

- (b) Both the numerator and the denominator of the above expression for the solution for y tend to zero as $\omega \to 1$ and are differentiable with respect to ω. According to L'Hôspital's rule,

$$\begin{aligned} \lim_{\omega\to 1} y(t) &= \lim_{\omega\to 1} \frac{\sin \omega t - \omega \sin t}{1-\omega^2} \\ &= \lim_{\omega\to 1} \frac{\frac{d}{d\omega}[\sin \omega t - \omega \sin t]}{\frac{d}{d\omega}[1-\omega^2]} \\ &= \lim_{\omega\to 1} \frac{t\cos \omega t - \sin t}{-2\omega} \\ &= \frac{t\cos t - \sin t}{-2} \\ &= \frac{1}{2}\left(-t\cos t + \sin t\right). \end{aligned}$$

- This limit is the solution of the 'resonant' initial value problem

$$\begin{aligned} &y'' + y = \sin t, \\ &y(0) = 0, \qquad y'(0) = 0. \end{aligned}$$

Note how the 'resonant' particular solution $t\cos t$ arose as a limit as $\omega \to 1$ of the 'nonresonant' particular solutions $\sin \omega t$.