Advanced Calculus
Math 25, Fall 2015
Midterm 1: Solutions

1. [20%] Give examples of the following. (No explanation or proof is required for this question).

(a) An onto function $f : \mathbb{N} \to \mathbb{N}$ that is not one-to-one.

(b) A sequence of bounded sets $A_n \subset \mathbb{R}$ such that $A_m \cap A_n = \emptyset$ for $m \neq n$ and $\bigcup_{n=1}^{\infty} A_n = \mathbb{R}$.

(c) A bounded set $A \subset \mathbb{R}$ of rational numbers such that $\inf A \in A$ and $\sup A \notin A$.

(d) A bounded sequence (x_n) of real numbers such that $x_n = 0$ for infinitely many $n \in \mathbb{N}$, $\liminf_{n\to\infty} x_n = -1$, and $\limsup_{n\to\infty} x_n = 1$.

Solution.

- (a) For example,

$$f(n) = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even,} \end{cases}$$

  so $f(1) = f(2) = 1$, $f(3) = f(4) = 2$, $f(5) = f(6) = 3$, and so on.

- (b) For example, take A_n to be the union of half-open intervals

$$A_n = (-n, -n+1] \cup [n-1, n),$$

  so

$$A_1 = (-1, 1), \quad A_2 = (-2, -1] \cup [1, 2), \quad A_3 = (-3, -2] \cup [2, 3),$$

  and so on.

- (c) For example,

$$A = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}.$$

  Then $\inf A = 0 \in A$ and $\sup A = 1 \notin A$.

- (d) For example, the sequence of alternating 0's, 1's and −1's:

  $$0, 1, -1, 0, 1, -1, 0, 1, -1, \ldots.$$

  Explicitly,

  $$x_n = \begin{cases} -1 & \text{if } n \equiv 0 \pmod{3}, \\ 0 & \text{if } n \equiv 1 \pmod{3}, \\ 1 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

2. [20%] Suppose that $x > 0$. Prove by induction that

$$1 + nx \leq (1+x)^n \qquad \text{for every } n \in \mathbb{N}. \tag{1}$$

Solution.

- Equation (1) is true for $n = 1$, since then $1 + nx = (1+x)^n$.
- Suppose (1) is true for some $n \in \mathbb{N}$. Then multiplying (1) by $1 + x > 0$, we get that
$$(1+nx)(1+x) \leq (1+x)^{n+1},$$
or
$$1 + (n+1)x + x^2 \leq (1+x)^{n+1}.$$
Since $x^2 > 0$, we have
$$1 + (n+1)x < 1 + (n+1)x + x^2 \leq (1+x)^{n+1},$$
and the result follows by induction. (In fact, we have strict inequality for $x > 0$ and $n \geq 2$).

Remark. Equation (1) is called Bernoulli's inequality. We proved it in class from the binomial theorem; the induction proof is perhaps a bit simpler.

3. [20%] Prove that if M is an upper bound of a set $A \subset \mathbb{R}$ and $M \in A$, then $M = \sup A$.

Solution.

- Since M is an upper bound of A, we just have to prove that it's a least upper bound. But if $M' < M$, then M' is not an upper bound of A since $M \in A$, so $M = \sup A$.

4. [20%] (a) State the definition of the convergence of a sequence (x_n) of real numbers to a limit x.
(b) For $n \in \mathbb{N}$, let

$$x_n = \frac{(1+n)^2}{1+n^2}.$$

Prove from the definition that $x_n \to 1$ as $n \to \infty$.

Solution.

- (a) $\lim_{n\to\infty} x_n = x$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n > N \text{ implies that } |x_n - x| < \epsilon.$$

(b) Let $\epsilon > 0$ and choose $N > 2/\epsilon$. Then if $n > N$, we have

$$\begin{aligned} \left| \frac{(1+n)^2}{1+n^2} - 1 \right| &= \left| \frac{1+2n+n^2}{1+n^2} - 1 \right| \\ &= \frac{2n}{1+n^2} \\ &< \frac{2}{n} \\ &< \frac{2}{N} \\ &< \epsilon, \end{aligned}$$

which proves the result.

5. [20%] Define a sequence (x_n) by $x_1 = x_2 = 1$ and

$$x_{n+1} = \frac{1}{2}x_n + \frac{2}{3}x_{n-1} \qquad \text{for } n \geq 2. \tag{2}$$

Prove that (x_n) cannot converge to a finite limit as $n \to \infty$.

Solution.

- First, we prove that $x_n \geq 1$ for every $n \in \mathbb{N}$. We take as our induction hypothesis that

  $$x_k \geq 1 \text{ for all } 1 \leq k \leq n.$$

  This is true for $n = 2$ since $x_1 = x_2 = 1$. Assume that it is true for some $n \geq 2$. Then

  $$x_{n+1} = \frac{1}{2}x_n + \frac{2}{3}x_{n-1} \geq \frac{1}{2} + \frac{2}{3} > 1,$$

  so the result follows by (strong) induction.

- Suppose for contradiction that $x_n \to x$ as $n \to \infty$. Then, since $x_n \geq 1$, the order property of limits implies that $x \geq 1$. Moreover, taking the limit of (2) as $n \to \infty$, we get from the algebraic properties of limits that

  $$x = \frac{1}{2}x + \frac{2}{3}x = \frac{7}{6}x,$$

  which implies that $x = 0$. It follows that (x_n) cannot converge.