Advanced Calculus
Math 25, Fall 2015
Midterm 2: Solutions

1. [25%] For each of the following series, determine (with proof) if it converges absolutely, converges conditionally, or diverges. (You can use any theorems proved in class.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{10^n}$ where (a_n) is a sequence of integers with $0 \leq a_n \leq 9$

(c) $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) = \log 2 + \log\frac{3}{2} + \log\frac{4}{3} + \dots$

(d) $\displaystyle\sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{(-1)^{n+1}}{\sqrt{n}}\right] = (1+1) + \left(\frac{1}{4} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{9} + \frac{1}{\sqrt{3}}\right) + \dots$

Solution.

- (a) $|\sin n/n^3| \leq 1/n^3$, so the series converges absolutely by comparison with the convergent p-series $\sum 1/n^3$.
- (b) $|a_n/10^n| \leq 9/10^n$, so the series converges absolutely by comparison with the convergent geometric series $\sum 9/10^n$.
- (c) This is a telescoping series, with partial sums

$$\begin{aligned}\sum_{n=1}^{N} \log\left(\frac{n+1}{n}\right) &= \sum_{n=1}^{N} [\log(n+1) - \log n] \\ &= \log(N+1) - \log 1 \\ &= \log(N+1).\end{aligned}$$

Since $\log(N+1) \to \infty$ as $N \to \infty$, the series diverges to ∞.

- (d) The series $\sum 1/n^2$ is a convergent p-series and the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

converges by the alternating series test, since $(1/\sqrt{n})$ is a decreasing sequence with limit zero. The term-by-term sum of the series therefore converges. The series does not converge absolutely (see below) so it converges conditionally

- One way to see that the series does not converge absolutely is to note that the term-by-term sum of two absolutely convergent series is absolutely convergent. Since

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = \sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{(-1)^{n+1}}{\sqrt{n}}\right] - \sum_{n=1}^{\infty} \frac{1}{n^2},$$

$\sum_{n=1}^{\infty} (-1)^{n+1}/\sqrt{n}$ is conditionally convergent and $\sum 1/n^2$ is absolutely convergent, the series in (d) cannot be absolutely convergent.

- Alternatively, For $n \geq 2$, we have

$$\left|\frac{1}{n^2} + \frac{(-1)^{n+1}}{\sqrt{n}}\right| \geq \frac{1}{\sqrt{n}} - \frac{1}{n^2} > \frac{1}{2\sqrt{n}},$$

so $\sum |1/n^2 + (-1)^{n+1}/\sqrt{n}|$ diverges to infinity by comparison with the divergent p-series $\sum 1/(2\sqrt{n})$.

2. [25%] Let

$$x_n = \sin\left(\frac{\pi\sqrt{n}}{4}\right).$$

(a) By what general theorem do you know that (x_n) has a convergent subsequence? Why?

(b) Give a subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = 1$. What is n_k?

(c) Give a subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = 0$. What is n_k?

(d) Does the sequence (x_n) converge? Why?

Solution.

- (a) We have $|x_n| \leq 1$ for every $n \in \mathbb{N}$, so the sequence is bounded, and the Bolzano-Weierstrass theorem implies that it has a convergent subsequence.

- (b) We have $\sin x = 1$ if $x = (2k+1/2)\pi$ where $k \in \mathbb{N}$, so we can choose n_k such that

  $$\frac{\pi\sqrt{n_k}}{4} = \left(2k + \frac{1}{2}\right)\pi,$$

  which gives

  $$n_k = (8k+2)^2.$$

- (c) We have $\sin x = 0$ if $x = 2k\pi$ where $k \in \mathbb{N}$, so we can choose n_k such that

  $$\frac{\pi\sqrt{n_k}}{4} = 2k\pi,$$

  which gives

  $$n_k = (8k)^2.$$

  (d) If a sequence converges, then every subsequence converges to the same limit. Since (x_n) has two subsequences converging to different limits, it diverges.

3. [25%] (a) State the Cauchy condition for the convergence of a series $\sum_{n=1}^{\infty} a_n$.
(b) Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers and $(a_{n_k})_{k=1}^{\infty}$ is a subsequence. If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, prove that the series $\sum_{k=1}^{\infty} a_{n_k}$ converges.
(c) Does the result in (b) remain true if $\sum_{n=1}^{\infty} a_n$ converges conditionally? Justify your answer.

Solution.

- (a) A series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > m > N$ implies that

$$\left| \sum_{k=m+1}^{n} a_k \right| < \epsilon.$$

- (b) Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} a_n$ converges absolutely, the series $\sum_{n=1}^{\infty} |a_n|$ converges, so it is Cauchy and there exists $N \in \mathbb{N}$ such that $n > m > N$ implies that

$$\sum_{k=m+1}^{n} |a_k| < \epsilon.$$

Choose $K \in \mathbb{N}$ such that $n_k > N$ for $k > K$. Then if $q > p > K$, we have

$$\sum_{k=p+1}^{q} |a_{n_k}| \le \sum_{k=n_p}^{n_q} |a_k| < \epsilon,$$

so $\sum_{k=1}^{\infty} |a_{n_k}|$ is Cauchy, and the series converges absolutely.

- (c) No, the result does not remain true. For example, the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges, but the subsequence of $a_n = (-1)^{n+1}/n$ with $n = 2k$ gives the divergent harmonic series

$$-\sum_{k=1}^{\infty} \frac{1}{2k}.$$

4. [25%] (a) State the comparison test for the convergence of a series.
(b) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with strictly positive terms $a_n > 0$ and $b_n > 0$. Suppose that the sequence (a_n/b_n) converges and the limit is strictly positive, so that $\lim_{n\to\infty} a_n/b_n = L > 0$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
(c) Does the series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

converge or diverge? HINT. You can use the fact that $\lim_{x\to 0} \sin x/x = 1$.

Solution.

- (a) Comparison test: If $|a_n| \leq b_n$ and the series $\sum b_n$ converges, then $\sum a_n$ converges absolutely.

- (b) Choose $\epsilon = L > 0$. Then there exists $N \in \mathbb{N}$ such that $n > N$ implies that

$$\left|\frac{a_n}{b_n} - L\right| < L.$$

  Then

$$\frac{a_n}{b_n} = \frac{a_n}{b_n} - L + L < 2L,$$

  so $0 < a_n < 2Lb_n$ for $n > N$. If $\sum b_n$ converges, then the comparison test with the convergent series $\sum 2Lb_n$ implies that $\sum a_n$ converges.

- Conversely, exchanging a_n and b_n, we have

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{L} > 0,$$

  so the convergence of $\sum a_n$ implies the convergence of $\sum b_n$.

- (c) Writing $x = 1/n$, we have

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} = 1,$$

  so the convergence of $\sum \sin(1/n)$ is the same as the convergence of $\sum 1/n$, and the series diverges.