Advanced Calculus
Math 25, Fall 2015
Sample Final Questions

1. Say if the following statements are true or false. If false, give a counter-example, if true give a brief explanation why (a complete proof is not required).
(a) If $a < b + 1/n$ for every $n \in \mathbb{N}$, then $a < b$.
(b) If $a \leq b + 1/n$ for every $n \in \mathbb{N}$, then $a \leq b$.
(c) $\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
(d) The sequence $(\cos n + \sin n)$ has a convergent subsequence.
(e) If $a_n \geq 0$ and $\sum a_n$ converges, then $\sum \sin a_n$ converges.
(f) If $F \subset \mathbb{R}$ is closed, then $\overline{F^\circ} = F$.

2. Suppose that $0 \leq a \leq 1$. Prove by induction that

$$(1 + a)^n \leq 1 + (2^n - 1)a \qquad \text{for every } n \in \mathbb{N}.$$

3. Let $A \subset \mathbb{R}$ be nonempty and bounded from above. Define

$$\begin{aligned} -A &= \{b \in \mathbb{R} : b = -a \text{ where } a \in A\} \\ B &= \{b \in \mathbb{R} : b \text{ is an upper bound for } A\}. \end{aligned}$$

Show that $\inf(-A) = -\sup A$ and $\inf B = \sup A$.

4. (a) State the definition of the convergence of a sequence (x_n) of real numbers to a limit $L \in \mathbb{R}$.
(b) Prove from the definition that

$$\lim_{n\to\infty} \frac{3n + 2}{7n - 5} = \frac{3}{7}.$$

4. Suppose that a sequence $(a_n)_{n=1}^{\infty}$ of real numbers does not converge to $L \in \mathbb{R}$. Prove that there exists $\epsilon_0 > 0$ and a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $|a_{n_k} - L| \geq \epsilon_0$ for every $k \in \mathbb{N}$.

5. (a) If (a_n) is a sequence of real numbers, state the definition of $a_n \to \infty$ as $n \to \infty$.

(b) Suppose that (a_n), (b_n) are two sequences of real numbers such that $a_n \to \infty$ and $b_n \to L$ as $n \to \infty$, where $0 < L < \infty$. Prove that $a_n b_n \to \infty$ as $n \to \infty$. Does this result remain true if $L = 0$?

6. (a) Suppose that $\{K_1, K_2, \dots, K_n\}$ is a finite collection of compact sets $K_i \subset \mathbb{R}$, and let

$$K = \bigcup_{i=1}^{n} K_i$$

Prove that K is compact. If $\{K_i : i \in \mathbb{N}\}$ is a countably infinite collection of compact sets, is $K = \bigcup_{i=1}^{\infty} K_i$ necessarily compact?

7. (a) Use the addition formula for cosines

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

to show that

$$\sin n = \frac{\cos(n - 1/2) - \cos(n + 1/2)}{2 \sin(1/2)}$$

(b) Show that

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}$$

converges.

Hint. You can use Abels's test: If the partial sums $A_N = \sum_{n=1}^{N} a_n$ form a bounded sequence, $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, and $b_n \to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

8. Let $A \subset \mathbb{R}$ and $\epsilon > 0$. Prove that the set

$$B = \{x \in \mathbb{R} : |x - y| < \epsilon \text{ for some } y \in A\}$$

is open.