Advanced Calculus

Math 25, Fall 2015

Sample Final Questions: Solutions

1. Say if the following statements are true or false. If false, give a counter-example, if true give a brief explanation why (a complete proof is not required).

(a) If $a < b + 1/n$ for every $n \in \mathbb{N}$, then $a < b$.

(b) If $a \leq b + 1/n$ for every $n \in \mathbb{N}$, then $a \leq b$.

(c) $\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

(d) The sequence $(\cos n + \sin n)$ has a convergent subsequence.

(e) If $a_n \geq 0$ and $\sum a_n$ converges, then $\sum \sin a_n$ converges.

(f) If $F \subset \mathbb{R}$ is closed, then $\overline{F^\circ} = F$.

Solution.

- (a) False. If $a = b$, then $a < b + 1/n$ for every $n \in \mathbb{N}$.
- (b) True. Proof of the contrapositive statement: If $a > b$, then $a-b > 0$ and, by the Archimedean principle, there exists $n \in \mathbb{N}$ such that $1/n < a - b$, so $a > b + 1/n$.
- (c) False. For example, if $a_n = (-1)^{n+1}$ and $b_n = (-1)^n$, then $a_n + b_n = 0$ and
$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1, \qquad \limsup_{n\to\infty}(a_n + b_n) = 0.$$
- (d) True. The sequence is bounded since
$$|\cos n + \sin n| \leq |\cos n| + |\sin n| \leq 2,$$
so the Bolzano-Weierstrass theorem implies that it has a convergent subsequence.
- (e) True. Since $|\sin x| \leq |x|$, we have $|\sin a_n| \leq a_n$, so $\sum \sin a_n$ converges (absolutely) by comparison with the convergent series $\sum a_n$.
- (f) False. For example, the set $F = \{0\}$ is closed, but $F^\circ = \emptyset$ and $\overline{F^\circ} = \emptyset \neq F$.

2. Suppose that $0 \le a \le 1$. Prove by induction that

$$(1+a)^n \le 1 + (2^n - 1)a \qquad \text{for every } n \in \mathbb{N}.$$

Solution.

- The result is true for $n = 1$, when $(1+a)^n = 1 + (2^n - 1)a$.
- Assume the result is true for some $n \in \mathbb{N}$. Then

$$\begin{aligned}(1+a)^{n+1} &= (1+a)^n(1+a) \\ &\le [1 + (2^n - 1)a](1+a) \\ &\le 1 + (2^n - 1)a + a + (2^n - 1)a^2.\end{aligned}$$

Since $0 \le a \le 1$, we have $a^2 \le a$, and therefore

$$\begin{aligned}(1+a)^{n+1} &\le 1 + (2^n - 1)a + a + (2^n - 1)a \\ &\le 1 + (2^{n+1} - 1)a,\end{aligned}$$

so the result holds for $n+1$. The result therefore holds for every $n \in \mathbb{N}$ by induction.

3. Let $A \subset \mathbb{R}$ be nonempty and bounded from above. Define

$$\begin{aligned} -A &= \{b \in \mathbb{R} : b = -a \text{ where } a \in A\} \\ B &= \{b \in \mathbb{R} : b \text{ is an upper bound for } A\}. \end{aligned}$$

Show that $\inf(-A) = -\sup A$ and $\inf B = \sup A$.

Solution.

- Let $M = \sup A$. Then M is an upper bound of A and $a \leq M$ for every $a \in A$. It follows that $-a \geq -M$, which shows that $-M$ is a lower bound of $-A$.

- If $m > -M$, then $-m < M$, so (since M is a least upper bound of A) the exists $a \in A$ such that $a > -m$. Then $-a < m$, so m is not a lower bound of $-A$. It follows that $-M$ is a greatest lower bound of $-A$, which proves that $\inf(-A) = -\sup A$.

- If b is an upper bound of A, then $b \geq M$, since $M = \sup A$ is the least upper bound, so M is a lower bound of B. Moreover, $M \in B$ since M is an upper bound of A. It follows that if $M' > M$, then M' is not a lower bound of B, so M is the greatest lower bound of B, which proves that $\inf B = \sup A$. (In fact, since $\sup A \in B$, it is the minimum of B.)

4. (a) State the definition of the convergence of a sequence (x_n) of real numbers to a limit $L \in \mathbb{R}$.
(b) Prove from the definition that

$$\lim_{n\to\infty} \frac{3n+2}{7n-5} = \frac{3}{7}.$$

Solution.

- (a) For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies that $|x_n - L| < \epsilon$.

- (b) Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that

  $$N \geq \frac{29}{49\epsilon}.$$

  If $n > N$, then

  $$\begin{aligned} \left|\frac{3n+2}{7n-5} - \frac{3}{7}\right| &= \left|\frac{29}{7(7n-5)}\right| \\ &< \frac{29}{49n} \\ &< \frac{29}{49N} \\ &< \epsilon, \end{aligned}$$

  which proves the convergence.

4. Suppose that a sequence $(a_n)_{n=1}^\infty$ of real numbers does not converge to $L \in \mathbb{R}$. Prove that there exists $\epsilon_0 > 0$ and a subsequence $(a_{n_k})_{k=1}^\infty$ such that $|a_{n_k} - L| \geq \epsilon_0$ for every $k \in \mathbb{N}$.

Solution.

- If (a_n) does not converge to L, then there exists $\epsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there exists $n > N$ with $|a_n - L| \geq \epsilon_0$. This condition is the negation of the definition for (a_n) to converge to L.

- Taking $N = 1$ in this condition, we get that there exists $n_1 \in \mathbb{N}$ such that $|a_{n_1} - L| \geq \epsilon_0$. Then, taking $N = n_1$, we get that there exists $n_2 > n_1$ such that $|a_{n_2} - L| \geq \epsilon_0$. Continuing in this way, given $n_k \in \mathbb{N}$, we get $n_{k+1} > n_k$ such that $|a_{n_{k+1}} - L| \geq \epsilon_0$, and then $(a_{n_k})_{k=1}^\infty$ is the required subsequence.

5. (a) If (a_n) is a sequence of real numbers, state the definition of $a_n \to \infty$ as $n \to \infty$.

(b) Suppose that (a_n), (b_n) are two sequences of real numbers such that $a_n \to \infty$ and $b_n \to L$ as $n \to \infty$, where $0 < L < \infty$. Prove that $a_n b_n \to \infty$ as $n \to \infty$. Does this result remain true if $L = 0$?

Solution.

- (a) For every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $n > N$ implies that $a_n > M$.

- (b) Let $M > 0$ be given (we can assume M is positive without loss of generality). Since $a_n \to \infty$ and $L > 0$, there exists $N_1 \in \mathbb{N}$ such that $n > N_1$ implies that
$$a_n > \frac{2M}{L}.$$
Also, since $b_n \to L$ and $L > 0$, there exists $N_2 \in \mathbb{N}$ such that $n > N_2$ implies that $|b_n - L| < L/2$, so
$$b_n = b_n - L + L > \frac{L}{2}.$$
Setting $N = \max\{N_1, N_2\}$, we find that for $n > N$
$$a_n b_n > \frac{2M}{L} \cdot \frac{L}{2} = M,$$
which proves that $a_n b_n \to \infty$ as $n \to \infty$.

- The result does not remain true if $L = 0$. For example, if $a_n = n$ and $b_n = 1/n$, then $a_n \to \infty$, $b_n \to 0$, and $a_n b_n \to 1$.

6. (a) Suppose that $\{K_1, K_2, \dots, K_n\}$ is a finite collection of compact sets $K_i \subset \mathbb{R}$, and let

$$K = \bigcup_{i=1}^{n} K_i$$

Prove that K is compact.

(b) If $\{K_i : i \in \mathbb{N}\}$ is a countably infinite collection of compact sets, is $K = \bigcup_{i=1}^{\infty} K_i$ necessarily compact?

Solution.

- (a) A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded. A finite union of closed sets is closed and a finite union of bounded sets is bounded (if $|x| \le M_i$ for every $x \in K_i$, then $|x| \le M$ for every $x \in \bigcup_{i=1}^{n} K_i$ where $M = \max\{M_1, M_2, \dots, M_n\}$). It follows that the finite union of compact sets is compact.

- (b) Countably infinite unions of closed sets need not be closed, and countably infinite unions of bounded sets need not be bounded, so countably infinite unions of compact sets need not be compact.

- For example,

$$K_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \qquad L_n = [-n, n]$$

  are compact sets, but

$$\bigcup_{n=1}^{\infty} K_n = (0, 1), \qquad \bigcup_{n=1}^{\infty} L_n = \mathbb{R}$$

  are not.

7. (a) Use the addition formula for cosines

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

to show that

$$\sin n = \frac{\cos(n - 1/2) - \cos(n + 1/2)}{2\sin(1/2)}$$

(b) Show that

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}$$

converges.

HINT. You can use Abels's test: If the partial sums $A_N = \sum_{n=1}^{N} a_n$ form a bounded sequence, $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, and $b_n \to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Solution.

- (a) Using the addition formula, we have

$$\begin{aligned} \cos(n - 1/2) - \cos(n + 1/2) &= \cos n \cos(1/2) + \sin n \sin(1/2) \\ &\qquad - [\cos n \cos(1/2) - \sin n \sin(1/2)] \\ &= 2 \sin n \sin(1/2), \end{aligned}$$

  and the result follows after we divide by $2\sin(1/2)$.

- (b) Let $a_n = \sin n$ and $b_n = 1/n$. Then (b_n) is a decreasing sequence that converges to 0 and

$$\begin{aligned} A_N &= \sum_{n=1}^{N} \sin n \\ &= \frac{1}{2\sin(1/2)} \sum_{n=1}^{N} [\cos(n - 1/2) - \cos(n + 1/2)] \\ &= \frac{1}{2\sin(1/2)} [\cos(1/2) - \cos(N + 1/2)] \end{aligned}$$

  is a telescoping sum. We have

$$|A_N| \leq \frac{1}{2\sin(1/2)} [|\cos(1/2)| + |\cos(N + 1/2)|] < \frac{1}{\sin(1/2)},$$

  so the sequence (A_N) is bounded and Abel's test implies that $\sum \sin n/n$ converges.

8. Let $A \subset \mathbb{R}$ and $\epsilon > 0$. Prove that the set

$$B = \{x \in \mathbb{R} : |x - y| < \epsilon \text{ for some } y \in A\}$$

is open.

Solution.

- Suppose $x \in B$, and choose $y \in A$ such that $|x - y| < \epsilon$. Let

  $$\delta = \epsilon - |x - y| > 0.$$

  If $|x' - x| < \delta$, then by the triangle inequality

  $$\begin{aligned} |x' - y| &= |x' - x + x - y| \\ &\leq |x' - x| + |x - y| \\ &< \delta + |x - y| \\ &< \epsilon. \end{aligned}$$

  It follows that $x' \in B$, so $(x - \delta, x + \delta) \subset B$, which proves that B is open.