Advanced Calculus
Math 25, Fall 2015
Sample Midterm Questions

1. If $f : X \to Y$ is a function and $A \subset X$, then we define $f(A) \subset Y$ by

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \in X\}.$$

(a) If $A, B \subset X$, prove that $f(A \cup B) = f(A) \cup f(B)$.
(b) Is $f(A \cap B) = f(A) \cap f(B)$?

2. Let $P(n, r)$ denote the rth element in the nth row of Pascal's triangle:

$$\begin{array}{lccccccccc}
n = 0: & & & & & 1 & & & & \\
n = 1: & & & & 1 & & 1 & & & \\
n = 2: & & & 1 & & 2 & & 1 & & \\
n = 3: & & 1 & & 3 & & 3 & & 1 & \\
n = 4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

where $0 \le r \le n$ e.g., $P(4, 2) = 6$. Then $P(n, 0) = P(n, n) = 1$ and

$$P(n + 1, r) = P(n, r - 1) + P(n, r)$$

for $1 \le r \le n$. Prove by induction that

$$P(n, r) = \frac{n!}{r!(n - r)!}.$$

3. (a) If $A, B \subset \mathbb{R}$ are nonempty sets that are bounded from above, prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.
(b) Is $\sup(A \cap B) = \min\{\sup A, \sup B\}$?

4. (a) State the definition of the convergence of a sequence (x_n) of real numbers to a limit x.
(b) For $n \in \mathbb{N}$, let

$$x_n = \frac{n \cos n + 3 \sin n}{n^2 + n - 10}.$$

Prove from the definition that $x_n \to 0$ as $n \to \infty$.

5. Suppose that (a_n), (b_n) are bounded sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Give an example of sequences where we have strict inequality in this equation.

6. Suppose that $A \subset \mathbb{R}$ is a nonempty set of real numbers that is bounded from above. Let $a \in A$ be an element of A and $b \in \mathbb{R}$ an upper bound of A. Construct two sequences (a_n), (b_n) of real numbers with $a_n \in A$ and b_n an upper bound of A as follows.

1. $a_1 = a$ and $b_1 = b$.
2. Given a_n and b_n, let $c_n = (a_n + b_n)/2$. (a) If c_n is an upper bound of A, then let $a_{n+1} = a_n$ and $b_{n+1} = c_n$. (b) If c_n is not an upper bound of A, then choose $a_{n+1} \in A$ such that $c_n \le a_{n+1} \le b_n$ and let $b_{n+1} = b_n$.

Prove that the sequences (a_n), (b_n) converge and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \sup A.$$

Hint. Note that $b_{n+1} - a_{n+1} \le (b_n - a_n)/2$.