Advanced Calculus

Math 25, Fall 2015

Sample Midterm Questions: Solutions

1. If $f : X \to Y$ is a function and $A \subset X$, then we define $f(A) \subset Y$ by

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \in X\}.$$

(a) If $A, B \subset X$, prove that $f(A \cup B) = f(A) \cup f(B)$.

(b) Is $f(A \cap B) = f(A) \cap f(B)$?

Solution.

- (a) If $x \in A \cup B$, then $x \in A$ or $x \in B$, so $f(x) \in f(A)$ or $f(x) \in f(B)$, meaning that $f(x) \in f(A) \cup f(B)$. It follows that $f(A \cup B) \subset f(A) \cup f(B)$.

- If $y \in f(A) \cup f(B)$, then $y \in f(A)$ or $y \in f(B)$, so $y = f(x)$ where $x \in A$ or $x \in B$, meaning that $x \in A \cup B$ and $y \in f(A \cup B)$. It follows that $f(A) \cup f(B) \subset f(A \cup B)$ and therefore $f(A \cup B) = f(A) \cup f(B)$.

- (b) This equality need not hold if f is not one-to-one. As a counter-example, let $X = \{a, b\}$ and $Y = \{c\}$ and define $f : X \to Y$ by $f(a) = f(b) = c$. If $A = \{a\}$ and $B = \{b\}$, then $A \cap B = \emptyset$, so $f(A \cap B) = \emptyset$, but $f(A) \cap f(B) = \{c\} \neq \emptyset$.

Remark. It is always the case that $f(A \cap B) \subset f(A) \cap f(B)$. Also,

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \qquad f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$

2. Let $P(n,r)$ denote the rth element in the nth row of Pascal's triangle:

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

where $0 \le r \le n$ e.g., $P(4,2)=6$. Then

$$\begin{aligned} P(n,0) = P(n,n) &= 1, \\ P(n+1,r) = P(n,r-1) + P(n,r) &\qquad \text{for } 1 \le r \le n. \end{aligned} \tag{1}$$

Prove by induction that

$$P(n,r) = \frac{n!}{r!(n-r)!}. \tag{2}$$

Solution.

- Equation (2) holds for $1 \le r \le n-1$ if $n=2$, since $P(2,1) = 2!/1!1!$.
- Suppose that (2) holds for $1 \le r \le n-1$ for some $n \ge 2$. Then (1) implies that for $2 \le r \le n-1$ we have

$$\begin{aligned} P(n+1,r) &= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} \\ &= \frac{n!}{(r-1)!(n-r)!}\left(\frac{1}{n-r+1} + \frac{1}{r}\right) \\ &= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right] \\ &= \frac{(n+1)!}{r!(n+1-r)!}. \end{aligned}$$

Also, if $r = 1, n$, we get from (1) that

$$\begin{aligned} P(n+1,1) &= 1 + \frac{n!}{1!(n-1)!} \\ &= n+1 \\ &= \frac{(n+1)}{1!n!}, \\ P(n+1,n) &= \frac{n!}{(n-1)!1!} + 1 \\ &= n+1 \\ &= \frac{(n+1)!}{n!1!}, \end{aligned}$$

so (2) holds for $n+1$ with $1 \leq r \leq n$. The result now follows by induction.

3. (a) If $A, B \subset \mathbb{R}$ are nonempty sets that are bounded from above, prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.
(b) Is $\sup(A \cap B) = \min\{\sup A, \sup B\}$?

Solution.

- (a) Let $M = \max\{\sup A, \sup B\}$. If $c \in A \cup B$, then $c \in A$ or $c \in B$, so $c \leq \sup A$ or $c \leq \sup B$. In either case, $c \leq M$, so M is an upper bound of $A \cup B$.

- If $M' < M$, then $M' < \sup A$ or $M' < \sup B$, and there exists $a \in A$ such that $a > M'$ or $b \in B$ such that $b > M'$. In either case, there exists $c \in A \cup B$ such that $c > M'$, so M' is not an upper bound of $A \cup B$. It follows that M is the least upper bound of $A \cup B$, which proves the result.

- (b) This is false. For example, if $A = \{0, 2\}$ and $B = \{0, 3\}$, then $\sup A = 2$ and $\sup B = 3$, so $\min\{\sup A, \sup B\} = 2$. On the other hand, $A \cap B = \{0\}$ so $\sup(A \cap B) = 0$.

4. (a) State the definition of the convergence of a sequence (x_n) of real numbers to a limit x.
(b) For $n \in \mathbb{N}$, let

$$x_n = \frac{n \cos n + 3 \sin n}{n^2 + n - 10}.$$

Prove from the definition that $x_n \to 0$ as $n \to \infty$.

Solution.

- (a) $x_n \to x$ as $n \to \infty$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies that $|x_n - x| < \epsilon$.

- (b) Given $\epsilon > 0$, choose $N = \max\{10, 2/\epsilon\}$. Then if $n > N$, we have

$$\begin{aligned} \left| \frac{n \cos n + 3 \sin n}{n^2 + n - 10} \right| &\le \frac{n + 3}{|n^2 + n - 10|} && \text{(since } |\cos n|, |\sin n| \le 1\text{)} \\ &< \frac{2n}{n^2} && \text{(since } n > 10\text{)} \\ &< \epsilon && \text{(since } n > 2/\epsilon\text{),} \end{aligned}$$

  which proves the result.

Remark. This value of N is not the optimal one, but we only have to show that there is some N that satisfies the definition.

5. Suppose that (a_n), (b_n) are bounded sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Give an example of sequences where we have strict inequality in this equation.

Solution.

- Let

$$\begin{aligned} y_n &= \sup\{a_k : k \ge n\}, \\ z_n &= \sup\{b_k : k \ge n\}, \\ w_n &= \sup\{a_k + b_k : k \ge n\}. \end{aligned}$$

  Then $w_n \le y_n + z_n$, since $y_n + z_n$ is an upper bound of $\{a_k + b_k : k \ge n\}$. The definition of the lim sup and the monotonicity and linearity of limits then implies that

$$\begin{aligned} \limsup_{n\to\infty}(a_n + b_n) &= \lim_{n\to\infty} w_n \\ &\le \lim_{n\to\infty}(y_n + z_n) \\ &\le \lim_{n\to\infty} y_n + \lim_{n\to\infty} z_n \\ &\le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n. \end{aligned}$$

- Let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1,$$

  but $a_n + b_n = 0$ and

$$\limsup_{n\to\infty}(a_n + b_n) = 0.$$

6. Suppose that $A \subset \mathbb{R}$ is a nonempty set of real numbers that is bounded from above. Let $a \in A$ be an element of A and $b \in \mathbb{R}$ an upper bound of A. Construct two sequences (a_n), (b_n) of real numbers with $a_n \in A$ and b_n an upper bound of A as follows.

1. $a_1 = a$ and $b_1 = b$.
2. Given a_n and b_n, let $c_n = (a_n + b_n)/2$. (a) If c_n is an upper bound of A, then let $a_{n+1} = a_n$ and $b_{n+1} = c_n$. (b) If c_n is not an upper bound of A, then choose $a_{n+1} \in A$ such that $c_n \le a_{n+1} \le b_n$ and let $b_{n+1} = b_n$.

Prove that the sequences (a_n), (b_n) converge and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \sup A.$$

Solution.

- By construction, $a_n \in A$ and b_n is an upper bound of A, so $a_n \le b_n$. It follows that $a_n \le c_n \le b_n$, and in both cases (a) and (b) we get $a_{n+1} \ge a_n$ and $b_{n+1} \le b_n$. Thus, (a_n) is an increasing sequence that is bounded above by b and (b_n) is a decreasing sequence that is bounded below by a.

- The convergence theorem for monotone sequences implies that the limits

  $$\lim_{n\to\infty} a_n = m, \qquad \lim_{n\to\infty} b_n = M$$

  both exist.

- In case (a), we have

  $$0 \le b_{n+1} - a_{n+1} = c_n - a_n = \frac{b_n - a_n}{2},$$

  and in case (b), we have

  $$0 \le b_{n+1} - a_{n+1} = b_n - a_{n+1} \le b_n - c_n = \frac{b_n - a_n}{2},$$

  so in either case

  $$0 \le b_{n+1} - a_{n+1} \le \frac{b_n - a_n}{2}.$$

- Taking the limit of this equation as $n \to \infty$, we get that

$$0 \leq M - m \leq \frac{M - m}{2},$$

which implies that $M - m = 0$.

- Finally, we show that the common limit $m = M$ of these sequences is equal to $\sup A$.

- First, if $x \in A$, then $x \leq b_n$ for every $n \in \mathbb{N}$, since b_n is an upper bound of A, so

$$x \leq \lim_{n\to\infty} b_n = M,$$

meaning that M is an upper bound of A.

- Second, if $\epsilon > 0$, then there exists $n \in \mathbb{N}$ such that $M - \epsilon < a_n \leq M$, since (a_n) is an increasing sequence that converges to M. Since $a_n \in A$, it follows that $M - \epsilon$ is not an upper bound of A, and therefore $M = \sup A$.

Remark. The assumption that every monotone increasing sequence of real numbers that is bounded from above has a limit provides an alternative formulation of the completeness axiom for $\mathbb{R}$. The preceding argument shows that this assumption implies that every set that is bounded from above has a supremum; we proved the reverse implication in class.