Advanced Calculus
Math 25, Fall 2015
Sample Midterm 2: Questions

1. For each of the following series, determine (with proof) if it converges absolutely, converges conditionally, or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+\sqrt{n}}$; (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n+\sqrt{n}}$;

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$; (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}(n+1)}$;

2. Say if the following statements are true or false and justify your answer.

(a) If every convergent subsequence of a sequence has the same limit, then the sequence converges.

(b) If a sequence has a divergent subsequence, then the sequence diverges.

(c) If $\sum a_n$ and $\sum (-1)^{n+1} a_n$ converge, then $\sum a_n$ converges absolutely.

3. (a) State the definition of a Cauchy sequence.

(b) Suppose that (x_n) is a sequence such that

$$|x_n - x_{n+1}| \le \frac{1}{2^n} \qquad \text{for every } n \in \mathbb{N}.$$

Prove that (x_n) converges.

(c) Suppose that (x_n) is a sequence such that

$$|x_n - x_{n+1}| \le \frac{1}{n} \qquad \text{for every } n \in \mathbb{N}.$$

Does it follow that (x_n) converges?

4. Prove the following statements. (You can use any standard properties or inequalities satisfied by $\cos x$ and $\sin x$.)

(a) If $\sum x_n$ converges then $\sum \cos x_n$ diverges.

(b) If $\sum x_n$ converges absolutely then $\sum \sin x_n$ converges.

5. (a) State the Bolzano-Weierstrass theorem.

(b) The nested interval property says that if (I_n) is a nested sequence of nonempty, closed, bounded intervals $I_n = [a_n, b_n]$ with $I_{n+1} \subset I_n$, then $\cap_{n=1}^{\infty} I_n$ is nonempty. Use the Bolzano-Weierstrass theorem to prove the nested interval property.