ADVANCED CALCULUS

Math 25, Fall 2015

Sample Midterm 2: Solutions

1. For each of the following series, determine (with proof) if it converges absolutely, converges conditionally, or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+\sqrt{n}}$; (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n+\sqrt{n}}$;

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$; (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}(n+1)}$;

Solution.

- (a) Since $\sqrt{n} \le n$ for $n \ge 1$, we have

$$\frac{1}{n+\sqrt{n}} \ge \frac{1}{2n},$$

  and the series diverges to ∞ by comparison with the harmonic series.

- (b) If $a_n = 1/(n+\sqrt{n})$, then (a_n) is a decreasing sequences with $a_n \to 0$ as $n \to \infty$, so the alternating series test implies that the series converges. From (a), the series does not converge absolutely, so it is conditionally convergent.

- (c) We have

$$\lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = 1,$$

  so the terms in the series do not have zero limit (in fact, the limit of the terms with alternating signs does not exist) and the series diverges.

- (d) This series is a telescoping series, and

$$\sum_{n=1}^{N} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = 1 - \frac{1}{\sqrt{N+1}} \to 1 \qquad \text{as } N \to \infty,$$

  so the series converges (to 1). Since the terms are positive, the series is absolutely convergent.

- (e) We have
$$\left|\frac{(-1)^{n+1}}{\sqrt{n}(n+1)}\right| < \frac{1}{n^{3/2}},$$
so the series converges absolutely by comparison with a convergent p-series with $p = 3/2 > 1$.

2. Say if the following statements are true or false and justify your answer.

(a) If every convergent subsequence of a sequence has the same limit, then the sequence converges.

(b) If a sequence has a divergent subsequence, then the sequence diverges.

(c) If $\sum a_n$ and $\sum (-1)^{n+1} a_n$ converge, then $\sum a_n$ converges absolutely.

Solution.

- (a) False. For example, in the case of the series (x_n)

$$1, 0, 2, 0, 3, 0, 4, 0, 5, 0, \ldots$$

with alternating increasing numbers $n \in \mathbb{N}$ and zeros,

$$x_{2n-1} = n, \qquad x_{2n} = 0,$$

the only convergent subsequences are the ones whose terms are eventually equal to 0 (all other subsequences are unbounded), so they have the same limit, but the sequence does not converge.

- (b) True. If a sequence converges, then every subsequence converges (to the same limit as the original sequence). The contrapositive statement says that if some subsequence diverges, then the sequence diverges.

- (c) False. For example, consider the series

$$1 + 0 - \frac{1}{2} + 0 + \frac{1}{3} + 0 - \frac{1}{4} + 0 + \frac{1}{5} + \ldots,$$

consisting of the alternating harmonic series with zero's inserted between every term. Explicitly,

$$a_{2n-1} = \frac{(-1)^{n+1}}{n}, \qquad a_{2n} = 0.$$

Then $(-1)^{n+1} a_n = a_n$ for every $n \in \mathbb{N}$, since $(-1)^{n+1} = 1$ if n is odd and $a_n = 0$ if n is even. Therefore both $\sum a_n$ and $\sum (-1)^{n+1} a_n$ converge because the alternating harmonic series converges, but the series does not converge absolutely since the harmonic series diverges.

3. (a) State the definition of a Cauchy sequence.
(b) Suppose that (x_n) is a sequence such that

$$|x_n - x_{n+1}| \leq \frac{1}{2^n} \qquad \text{for every } n \in \mathbb{N}.$$

Prove that (x_n) converges.
(c) Suppose that (x_n) is a sequence such that

$$|x_n - x_{n+1}| \leq \frac{1}{n} \qquad \text{for every } n \in \mathbb{N}.$$

Does it follow that (x_n) converges?

Solution.

- (a) A sequence (x_n) is Cauchy if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n > N$ implies that $|x_m - x_n| < \epsilon$.

- (b) For $n > m$, we have the telescoping sum

$$\begin{aligned} x_n - x_m &= (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_{m+1} - x_m) \\ &= \sum_{k=m+1}^{n} (x_k - x_{k-1}). \end{aligned}$$

  It follows that

$$|x_n - x_m| \leq \sum_{k=m+1}^{n} |x_k - x_{k-1}| \leq \sum_{k=m+1}^{n} \frac{1}{2^k}.$$

  Let $\epsilon > 0$. The geometric series $\sum 1/2^n$ converges, so by the Cauchy condition for series there exists $N \in \mathbb{N}$ such that $n > m > N$ implies that

$$\sum_{k=m+1}^{n} \frac{1}{2^k} < \epsilon.$$

  It follows that $m, n > N$ implies that $|x_m - x_n| < \epsilon$, so the sequence (x_n) is a Cauchy sequence and therefore it converges.

- (c) It does not follow that (x_n) converges. The previous proof fails because the harmonic series $\sum 1/n$ diverges. To give an explicit counterexample, let
$$x_n = \sum_{k=1}^{n} \frac{1}{k}.$$
Then
$$|x_n - x_{n+1}| = \frac{1}{n+1} < \frac{1}{n}$$
but (x_n) diverges to ∞ as $n \to \infty$.

4. Prove the following statements. (You can use any standard properties or inequalities satisfied by $\cos x$ and $\sin x$.)

(a) If $\sum x_n$ converges then $\sum \cos x_n$ diverges.

(b) If $\sum x_n$ converges absolutely then $\sum \sin x_n$ converges.

Solution.

- (a) If $\sum x_n$ converges, then $x_n \to 0$ as $n \to \infty$, so $\cos x_n \to 1$ as $n \to \infty$. Since the terms in the series $\sum \cos x_n$ have non-zero limit, the series diverges.
- (b) We have the inequality $|\sin x| \leq |x|$ for every $x \in \mathbb{R}$, so $\sum \sin x_n$ converges (absolutely) by comparison with the series $\sum |x_n|$, which converges since $\sum x_n$ converges absolutely.

5. (a) State the Bolzano-Weierstrass theorem.

(b) The nested interval property says that if (I_n) is a nested sequence of nonempty, closed, bounded intervals $I_n = [a_n, b_n]$ with $I_{n+1} \subset I_n$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty. Use the Bolzano-Weierstrass theorem to prove the nested interval property.

Solution.

- (a) Bolzano-Weierstrass theorem: Every bounded sequence of real numbers has a convergent subsequence.

- (b) For each $n \in \mathbb{N}$, choose $x_n \in I_n$. Since $x_n \in I_1$ for every $n \in \mathbb{N}$, the sequence (x_n) is bounded, and it follows from the Bolzano-Weierstrass theorem that it has a convergent subsequence (x_{n_k}). If $x_{n_k} \to x$ as $k \to \infty$, then we claim that $x \in \bigcap_{n=1}^{\infty} I_n$, so the intersection is nonempty.

- To prove the claim, note that for every $N \in \mathbb{N}$, we have $x_{n_k} \in I_N$ for all sufficiently large $k \in \mathbb{N}$ (such that $n_k \geq N$). It follows that $a_N \leq x_{n_k} \leq b_N$. Taking the limit of this inequality as $k \to \infty$ and using the monotonicity property of limits, we get that $a_N \leq x \leq b_N$. Thus, $x \in I_N$ for every $N \in \mathbb{N}$.