Measure Theory

John K. Hunter

Department of Mathematics, University of California at Davis

Abstract. These are some brief notes on measure theory, concentrating on Lebesgue measure on $\mathbb{R}^n$. Some missing topics I would have liked to have included had time permitted are: the change of variable formula for the Lebesgue integral on $\mathbb{R}^n$; absolutely continuous functions and functions of bounded variation of a single variable and their connection with Lebesgue-Stieltjes measures on $\mathbb{R}$; Radon measures on $\mathbb{R}^n$, and other locally compact Hausdorff topological spaces, and the Riesz representation theorem for bounded linear functionals on spaces of continuous functions; and other examples of measures, including k-dimensional Hausdorff measure in $\mathbb{R}^n$, Wiener measure and Brownian motion, and Haar measure on topological groups. All these topics can be found in the references.

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