Asymptotic Analysis and Singular Perturbation Theory

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Chapter 1

Introduction

In this chapter, we describe the aims of perturbation theory in general terms, and give some simple illustrative examples of perturbation problems. Some texts and references on perturbation theory are [8], [9], and [13].

1.1 Perturbation theory

Consider a problem

$$P^\varepsilon(x) = 0 \tag{1.1}$$

depending on a small, real-valued parameter ε that simplifies in some way when $\varepsilon = 0$ (for example, it is linear or exactly solvable). The aim of perturbation theory is to determine the behavior of the solution $x = x^\varepsilon$ of (1.1) as $\varepsilon \to 0$. The use of a small parameter here is simply for definiteness; for example, a problem depending on a large parameter ω can be rewritten as one depending on a small parameter $\varepsilon = 1/\omega$.

The focus of these notes is on perturbation problems involving differential equations, but perturbation theory and asymptotic analysis apply to a broad class of problems. In some cases, we may have an explicit expression for x^ε, such as an integral representation, and want to obtain its behavior in the limit $\varepsilon \to 0$.

1.1.1 *Asymptotic solutions*

The first goal of perturbation theory is to construct a formal asymptotic solution of (1.1) that satisfies the equation up to a small error. For example, for each $N \in \mathbb{N}$, we may be able to find an asymptotic solution x_N^ε such that

$$P^\varepsilon\left(x_N^\varepsilon\right) = O(\varepsilon^{N+1}),$$

where $O(\varepsilon^n)$ denotes a term of the the order ε^n. This notation will be made precise in Chapter 2.

Once we have constructed such an asymptotic solution, we would like to know that there is an exact solution $x = x^\varepsilon$ of (1.1) that is close to the asymptotic solution when ε is small; for example, a solution such that

$$x^\varepsilon = x_N^\varepsilon + O(\varepsilon^{N+1}).$$

This is the case if a small error in the equation leads to a small error in the solution. For example, we can establish such a result if we have a stability estimate of the form

$$|x - y| \leq C\,|P^\varepsilon(x) - P^\varepsilon(y)|$$

where C is a constant independent of ε, and $|\cdot|$ denotes appropriate norms. Such an estimate depends on the properties of P^ε and may be difficult to obtain, especially for nonlinear problems. In these notes we will focus on methods for the construction of asymptotic solutions, and we will not discuss in detail the existence of solutions close to the asymptotic solution.

1.1.2 *Regular and singular perturbation problems*

It is useful to make an imprecise distinction between *regular perturbation problems* and *singular perturbation problems*. A regular perturbation problem is one for which the perturbed problem for small, nonzero values of ε is qualitatively the same as the unperturbed problem for $\varepsilon = 0$. One typically obtains a convergent expansion of the solution with respect to ε, consisting of the unperturbed solution and higher-order corrections. A singular perturbation problem is one for which the perturbed problem is qualitatively different from the unperturbed problem. One typically obtains an asymptotic, but possibly divergent, expansion of the solution, which depends singularly on the parameter ε.

Although singular perturbation problems may appear atypical, they are the most interesting problems to study because they allow one to understand qualitatively new phenomena.

The solutions of singular perturbation problems involving differential equations often depend on several widely different length or time scales. Such problems can be divided into two broad classes: layer problems, treated using the method of matched asymptotic expansions (MMAE); and multiple-scale problems, treated by the method of multiple scales (MMS). Prandtl's boundary layer theory for the high-Reynolds flow of a viscous fluid over a solid body is an example of a boundary layer problem, and the semi-classical limit of quantum mechanics is an example of a multiple-scale problem.

We will begin by illustrating some basic issues in perturbation theory with simple algebraic equations.

1.2 Algebraic equations

The first two examples illustrate the distinction between regular and singular perturbation problems.

Example 1.1 Consider the cubic equation

$$x^3 - x + \varepsilon = 0. \tag{1.2}$$

We look for a solution of the form

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3). \tag{1.3}$$

Using this expansion in the equation, expanding, and equating coefficients of ε^n to zero, we get

$$\begin{aligned} &x_0^3 - x_0 = 0, \\ &3x_0^2 x_1 - x_1 + 1 = 0, \\ &3x_0 x_2 - x_2 + 3x_0 x_1^2 = 0. \end{aligned}$$

Note that we obtain a nonlinear equation for the leading order solution x_0, and nonhomogeneous linearized equations for the higher order corrections x_1, x_2,.... This structure is typical of many perturbation problems.

Solving the leading-order perturbation equation, we obtain the three roots

$$x_0 = 0, \pm 1.$$

Solving the first-order perturbation equation, we find that

$$x_1 = \frac{1}{1 - 3x_0^2}.$$

The corresponding solutions are

$$x = \varepsilon + O(\varepsilon^2), \qquad x = \pm 1 - \frac{1}{2}\varepsilon + O(\varepsilon^2).$$

Continuing in this way, we can obtain a convergent power series expansion about $\varepsilon = 0$ for each of the three distinct roots of (1.2). This result is typical of regular perturbation problems.

An alternative — but equivalent — method to obtain the perturbation series is to use the Taylor expansion

$$x(\varepsilon) = x(0) + \dot{x}(0)\varepsilon + \frac{1}{2!}\ddot{x}(0)\varepsilon^2 + \dots,$$

where the dot denotes a derivative with respect to ε. To compute the coefficients, we repeatedly differentiate the equation with respect to ε and set $\varepsilon = 0$ in the result.

For example, setting $\varepsilon = 0$ in (1.2), and solving the resulting equation for $x(0)$, we get $x(0) = 0, \pm 1$. Differentiating (1.2) with respect to ε, we get

$$3x^2\dot{x} - \dot{x} + 1 = 0.$$

Setting $\varepsilon = 0$ and solving for $\dot{x}(0)$, we get the same answer as before.

Example 1.2 Consider the cubic equation

$$\varepsilon x^3 - x + 1 = 0. \tag{1.4}$$

Using (1.3) in (1.4), expanding, and equating coefficents of ε^n to zero, we get

$$\begin{aligned} -x_0 + 1 &= 0, \\ -x_1 + x_0^3 &= 0, \\ -x_2 + 3x_0^2 x_1 &= 0. \end{aligned}$$

Solving these equations, we find that $x_0 = 1$, $x_1 = 1$, $\ldots$, and hence

$$x(\varepsilon) = 1 + \varepsilon + O(\varepsilon^2). \tag{1.5}$$

We only obtain one solution because the cubic equation (1.4) degenerates to a linear equation at $\varepsilon = 0$. We missed the other two solutions because they approach infinity as $\varepsilon \to 0$. A change in the qualitative nature of the problem at the unperturbed value $\varepsilon = 0$ is typical of singular perturbation problems.

To find the other solutions, we introduce a rescaled variable y, where

$$x(\varepsilon) = \frac{1}{\delta(\varepsilon)} y(\varepsilon),$$

and $y = O(1)$ as $\varepsilon \to 0$. The scaling factor δ is to be determined. Using this equation in (1.4), we find that

$$\frac{\varepsilon}{\delta^3} y^3 - \frac{1}{\delta} y + 1 = 0. \tag{1.6}$$

In order to obtain a nontrivial solution, we require that at least two leading-order terms in this equation have the same order of magnitude. This is called the principle of *dominant balance*.

Balancing the first two terms, we find that*

$$\frac{\varepsilon}{\delta^3} = \frac{1}{\delta},$$

which implies that $\delta = \varepsilon^{1/2}$. The first two terms in (1.4) are then $O(\varepsilon^{-1/2})$, and the third term is $O(1)$, which is smaller. With this choice of δ, equation (1.6) becomes

$$y^3 - y + \varepsilon^{1/2} = 0.$$

*Nonzero constant factors can be absorbed into y.

Solving this equation in the same way as (1.2), we get the nonzero solutions

$$y = \pm 1 - \frac{1}{2}\varepsilon^{1/2} + O(\varepsilon).$$

The corresponding solutions for x are

$$x = \pm\frac{1}{\varepsilon^{1/2}} - \frac{1}{2} + O\left(\varepsilon^{1/2}\right).$$

The dominant balance argument illustrated here is useful in many perturbation problems. The corresponding limit, $\varepsilon \to 0$ with $x(\varepsilon) = O(\varepsilon^{-1/2})$, is called a *distinguished limit*.

There are two other two-term balances in (1.6). Balancing the second and third terms, we find that

$$\frac{1}{\delta} = 1$$

or $\delta = 1$. The first term is then $O(\varepsilon)$, so it is smaller than the other two terms. This dominant balance gives the solution in (1.5). Balancing the first and third terms, we find that

$$\frac{\varepsilon}{\delta^3} = 1,$$

or $\delta = \varepsilon^{1/3}$. In this case, the first and third terms are $O(1)$, but the second term is $O(\varepsilon^{-1/3})$. Thus, it is larger than the terms that balance, so we do not obtain a dominant balance or any new solutions.

In this example, no three-term dominant balance is possible as $\varepsilon \to 0$, but this can occur in other problems.

Example 1.3 A famous example of the effect of a perturbation on the solutions of a polynomial is Wilkinson's polynomial (1964),

$$(x-1)(x-2)\dots(x-20) = \varepsilon x^{19}.$$

The perturbation has a large effect on the roots even for small values of ε.

The next two examples illustrate some other features of perturbation theory.

Example 1.4 Consider the quadratic equation

$$(1-\varepsilon)x^2 - 2x + 1 = 0.$$

Suppose we look for a straightforward power series expansion of the form

$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2).$$

We find that

$$x_0^2 - 2x_0 + 1 = 0,$$
$$2(x_0 - 1)x_1 = x_0^2.$$

Solving the first equation, we get $x_0 = 1$. The second equation then becomes $0 = 1$. It follows that there is no solution of the assumed form.

This difficulty arises because $x = 1$ is a repeated root of the unperturbed problem. As a result, the solution

$$x = \frac{1 \pm \varepsilon^{1/2}}{1 - \varepsilon}$$

does not have a power series expansion in ε, but depends on $\varepsilon^{1/2}$. An expansion

$$x = x_0 + \varepsilon^{1/2} x_1 + \varepsilon x_2 + O(\varepsilon^{3/2})$$

leads to the equations $x_0 = 1$, $x_1^2 = 1$, or

$$x = 1 \pm \varepsilon^{1/2} + O(\varepsilon)$$

in agreement with the exact solution.

Example 1.5 Consider the transcendental equation

$$xe^{-x} = \varepsilon. \tag{1.7}$$

As $\varepsilon \to 0^+$, there are two possibilities:

(a) $x \to 0$, which implies that $x = \varepsilon + \varepsilon^2 + O(\varepsilon^2)$;
(b) $e^{-x} \to 0$, when $x \to \infty$.

In the second case, x must be close to $\log 1/\varepsilon$.

To obtain an asymptotic expansion for the solution, we solve the equation iteratively using the idea that e^{-x} varies much more rapidly than x as $x \to 0$. Rewriting (1.7) as $e^{-x} = \varepsilon/x$ and taking logarithms, we get the equivalent equation

$$x = \log x + \log \frac{1}{\varepsilon}.$$

Thus solutions are fixed points of the function

$$f(x) = \log x + \log \frac{1}{\varepsilon}.$$

We then define iterates x_n, $n \in \mathbb{N}$, by

$$x_{n+1} = \log x_n + \log \frac{1}{\varepsilon},$$
$$x_1 = \log \frac{1}{\varepsilon}.$$

Defining

$$L_1 = \log \frac{1}{\varepsilon}, \qquad L_2 = \log \left(\log \frac{1}{\varepsilon} \right),$$

we find that

$$\begin{aligned} x_2 &= L_1 + L_2, \\ x_3 &= L_1 + \log(L_1 + L_2) \\ &= L_1 + L_2 + \frac{L_2}{L_1} + O\left(\left(\frac{L_2}{L_1}\right)^2\right). \end{aligned}$$

At higher orders, terms involving

$$L_3 = \log\left(\log\left(\log\frac{1}{\varepsilon}\right)\right),$$

and so on, appear.

The form of this expansion would be difficult to guess without using an iterative method. Note, however, that the successive terms in this asymptotic expansion converge very slowly as $\varepsilon \to 0$. For example, although $L_2/L_1 \to 0$ as $\varepsilon \to 0$, when $\varepsilon = 0.1$, $L_1 \approx 36$, $L_2 \approx 12$; and when $\varepsilon = 10^{-5}$, $L_1 \approx 19$, $L_2 \approx 1$.

1.3 Eigenvalue problems

Spectral perturbation theory studies how the spectrum of an operator is perturbed when the operator is perturbed. In general, this question is a difficult one, and subtle phenomena may occur, especially in connection with the behavior of the continuous spectrum of the operators. Here, we consider the simplest case of the perturbation in an eigenvalue.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $A^\varepsilon : \mathcal{D}(A^\varepsilon) \subset \mathcal{H} \to \mathcal{H}$ a linear operator in $\mathcal{H}$, with domain $\mathcal{D}(A^\varepsilon)$, depending smoothly on a real parameter ε. We assume that:

(a) A^ε is self-adjoint, so that

$$\langle x, A^\varepsilon y \rangle = \langle A^\varepsilon x, y \rangle \qquad \text{for all } x, y \in \mathcal{D}(A^\varepsilon);$$

(b) A^ε has a smooth branch of simple eigenvalues $\lambda^\varepsilon \in \mathbb{R}$ with eigenvectors $x^\varepsilon \in \mathcal{H}$, meaning that

$$A^\varepsilon x^\varepsilon = \lambda^\varepsilon x^\varepsilon. \tag{1.8}$$

We will compute the perturbation in the eigenvalue from its value at $\varepsilon = 0$ when ε is small but nonzero.

A concrete example is the perturbation in the eigenvalues of a symmetric matrix. In that case, we have $\mathcal{H} = \mathbb{R}^n$ with the Euclidean inner product

$$\langle x, y \rangle = x^T y,$$

and $A^\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with an $n \times n$ symmetric matrix (a_{ij}^ε). The perturbation in the eigenvalues of a Hermitian matrix corresponds to $\mathcal{H} = \mathbb{C}^n$ with inner product $\langle x, y\rangle = \overline{x}^T y$. As we illustrate below with the Schrödinger equation of quantum mechanics, spectral problems for differential equations can be formulated in terms of unbounded operators acting in infinite-dimensional Hilbert spaces.

We use the expansions

$$\begin{aligned} A^\varepsilon &= A_0 + \varepsilon A_1 + \ldots + \varepsilon^n A_n + \ldots, \\ x^\varepsilon &= x_0 + \varepsilon x_1 + \ldots + \varepsilon^n x_n + \ldots, \\ \lambda^\varepsilon &= \lambda_0 + \varepsilon \lambda_1 + \ldots + \varepsilon^n \lambda_n + \ldots \end{aligned}$$

in the eigenvalue problem (1.8), equate coefficients of ε^n, and rearrange the result. We find that

$$(A_0 - \lambda_0 I)\, x_0 = 0, \tag{1.9}$$

$$(A_0 - \lambda_0 I)\, x_1 = -A_1 x_0 + \lambda_1 x_0, \tag{1.10}$$

$$(A_0 - \lambda_0 I)\, x_n = \sum_{i=1}^{n} \left\{ -A_i x_{n-i} + \lambda_i x_{n-i} \right\}. \tag{1.11}$$

Assuming that $x_0 \neq 0$, equation (1.9) implies that λ_0 is an eigenvalue of A_0 and x_0 is an eigenvector. Equation (1.10) is then a singular equation for x_1. The following proposition gives a simple, but fundamental, solvability condition for this equation.

Proposition 1.6 Suppose that A is a self-adjoint operator acting in a Hilbert space $\mathcal{H}$ and $\lambda \in \mathbb{R}$. If $z \in \mathcal{H}$, a necessary condition for the existence of a solution $y \in \mathcal{H}$ of the equation

$$(A - \lambda I)\, y = z \tag{1.12}$$

is that

$$\langle x, z\rangle = 0,$$

for every eigenvector x of A with eigenvalue λ.

Proof. Suppose $z \in \mathcal{H}$ and y is a solution of (1.12). If x is an eigenvector of A, then using (1.12) and the self-adjointness of $A - \lambda I$, we find that

$$\begin{aligned} \langle x, z\rangle &= \langle x, (A - \lambda I)\, y\rangle \\ &= \langle (A - \lambda I)\, x, y\rangle \\ &= 0. \end{aligned}$$

□

In many cases, the necesary solvability condition in this proposition is also sufficient, and then we say that $A - \lambda I$ satisfies the Fredholm alternative; for example, this is true in the finite-dimensional case, or when A is an elliptic partial differential operator.

Since A_0 is self-adjoint and λ_0 is a simple eigenvalue with eigenvector x_0, equation (1.12) it is solvable for x_1 only if the right hand side is orthogonal to x_0, which implies that

$$\lambda_1 = \frac{\langle x_0, A_1 x_0 \rangle}{\langle x_0, x_0 \rangle}.$$

This equation gives the leading order perturbation in the eigenvalue, and is the most important result of the expansion.

Assuming that the necessary solvability condition in the proposition is sufficient, we can then solve (1.10) for x_1. A solution for x_1 is not unique, since we can add to it an arbitrary scalar multiple of x_0. This nonuniqueness is a consequence of the fact that if x^ε is an eigenvector of A^ε, then $c^\varepsilon x^\varepsilon$ is also a solution for any scalar c^ε. If

$$c^\varepsilon = 1 + \varepsilon c_1 + O(\varepsilon^2)$$

then

$$c^\varepsilon x^\varepsilon = x_0 + \varepsilon \left(x_1 + c_1 x_0 \right) + O(\varepsilon^2).$$

Thus, the addition of $c_1 x_0$ to x_1 corresponds to a rescaling of the eigenvector by a factor that is close to one.

This expansion can be continued to any order. The solvability condition for (1.11) determines λ_n, and the equation may then be solved for x_n, up to an arbitrary vector $c_n x_0$. The appearance of singular problems, and the need to impose solvabilty conditions at each order which determine parameters in the expansion and allow for the solution of higher order corrections, is a typical structure of many pertubation problems.

1.3.1 *Quantum mechanics*

One application of this expansion is in quantum mechanics, where it can be used to compute the change in the energy levels of a system caused by a perturbation in its Hamiltonian.

The Schrödinger equation of quantum mechanics is

$$i\hbar \psi_t = H\psi.$$

Here t denotes time and $\hbar$ is Planck's constant. The wavefunction $\psi(t)$ takes values in a Hilbert space $\mathcal{H}$, and H is a self-adjoint linear operator acting in $\mathcal{H}$ with the dimensions of energy, called the Hamiltonian.

Energy eigenstates are wavefunctions of the form

$$\psi(t) = e^{-iEt/\hbar}\varphi,$$

where $\varphi \in \mathcal{H}$ and $E \in \mathbb{R}$. It follows from the Schrödinger equation that

$$H\varphi = E\varphi.$$

Hence E is an eigenvalue of H and φ is an eigenvector. One of Schrödinger's motivations for introducing his equation was that eigenvalue problems led to the experimentally observed discrete energy levels of atoms.

Now suppose that the Hamiltonian

$$H^\varepsilon = H_0 + \varepsilon H_1 + O(\varepsilon^2)$$

depends smoothly on a parameter ε. Then, rewriting the previous result, we find that the corresponding simple energy eigenvalues (assuming they exist) have the expansion

$$E^\varepsilon = E_0 + \varepsilon \frac{\langle \varphi_0, H_1\varphi_0\rangle}{\langle \varphi_0, \varphi_0\rangle} + O(\varepsilon^2)$$

where φ_0 is an eigenvector of H_0.

For example, the Schrödinger equation that describes a particle of mass m moving in $\mathbb{R}^d$ under the influence of a conservative force field with potential $V : \mathbb{R}^d \to \mathbb{R}$ is

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V\psi.$$

Here, the wavefunction $\psi(x,t)$ is a function of a space variable $x \in \mathbb{R}^d$ and time $t \in \mathbb{R}$. At fixed time t, we have $\psi(\cdot,t) \in L^2(\mathbb{R}^d)$, where

$$L^2(\mathbb{R}^d) = \left\{u : \mathbb{R}^d \to \mathbb{C} \mid u \text{ is measurable and } \textstyle\int_{\mathbb{R}^d} |u|^2\, dx < \infty\right\}$$

is the Hilbert space of square-integrable functions with inner-product

$$\langle u, v\rangle = \int_{\mathbb{R}^d} \overline{u}(x)v(x)\, dx.$$

The Hamiltonian operator $H : \mathcal{D}(H) \subset \mathcal{H} \to \mathcal{H}$, with domain $\mathcal{D}(H)$, is given by

$$H = -\frac{\hbar^2}{2m}\Delta + V.$$

If u, v are smooth functions that decay sufficiently rapidly at infinity, then Green's theorem implies that

$$\begin{aligned}\langle u, Hv\rangle &= \int_{\mathbb{R}^d} \overline{u}\left(-\frac{\hbar^2}{2m}\Delta v + Vv\right) dx \\ &= \int_{\mathbb{R}^d} \left\{\frac{\hbar^2}{2m}\nabla\cdot(v\nabla\overline{u} - \overline{u}\nabla v) - \frac{\hbar^2}{2m}(\Delta\overline{u})v + V\overline{u}v\right\} dx\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \overline{\left(-\frac{\hbar^2}{2m}\Delta u + Vu\right)} v \, dx \\
&= \langle Hu, v\rangle.
\end{aligned}
$$

Thus, this operator is formally self-adjoint. Under suitable conditions on the potential V, the operator can be shown to be self-adjoint with respect to an appropriately chosen domain.

Now suppose that the potential V^ε depends on a parameter ε, and has the expansion

$$V^\varepsilon(x) = V_0(x) + \varepsilon V_1(x) + O(\varepsilon^2).$$

The perturbation in a simple energy eigenvalue

$$E^\varepsilon = E_0 + \varepsilon E_1 + O(\varepsilon^2),$$

assuming one exists, is given by

$$E_1 = \frac{\int_{\mathbb{R}^d} V_1(x)|\varphi_0(x)|^2 \, dx}{\int_{\mathbb{R}^d} |\varphi_0(x)|^2 \, dx},$$

where $\varphi_0 \in L^2(\mathbb{R}^d)$ is an unperturbed energy eigenfunction that satisfies

$$-\frac{\hbar^2}{2m}\Delta\varphi_0 + V_0\varphi_0 = E_0\varphi_0.$$

Example 1.7 The one-dimensional simple harmonic oscillator has potential

$$V_0(x) = \frac{1}{2}kx^2.$$

The eigenvalue problem

$$-\frac{\hbar^2}{2m}\varphi'' + \frac{1}{2}kx^2\varphi = E\varphi, \qquad \varphi \in L^2(\mathbb{R})$$

is exactly solvable. The energy eigenvalues are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2, \ldots,$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

is the frequency of the corresponding classical oscillator. The eigenfunctions are

$$\varphi_n(x) = H_n(\alpha x)e^{-\alpha^2 x^2/2},$$

where H_n is the nth Hermite polynomial,

$$H_n(\xi) = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2},$$

and the constant α, with dimensions of 1/length, is given by

$$\alpha^2 = \frac{\sqrt{mk}}{\hbar}.$$

The energy levels E_n^ε of a slightly anharmonic oscillator with potential

$$V^\varepsilon(x) = \frac{1}{2}kx^2 + \varepsilon\frac{k}{\alpha^2}W(\alpha x) + O(\varepsilon^2) \qquad \text{as } \varepsilon \to 0^+$$

where $\varepsilon > 0$ have the asymptotic behavior

$$E_n^\varepsilon = \hbar\omega\left\{n + \frac{1}{2} + \varepsilon\Delta_n + O(\varepsilon^2)\right\} \qquad \text{as } \varepsilon \to 0^+,$$

where

$$\Delta_n = \frac{\int W(\xi)H_n^2(\xi)e^{-\xi^2}\,d\xi}{\int H_n^2(\xi)e^{-\xi^2}\,d\xi}.$$

For an extensive and rigorous discussion of spectral perturbation theory for linear operators, see [11].

1.4 Nondimensionalization

The numerical value of any quantity in a mathematical model is measured with respect to a system of units (for example, meters in a mechanical model, or dollars in a financial model). The units used to measure a quantity are arbitrary, and a change in the system of units (for example, to feet or yen, at a fixed exchange rate) cannot change the predictions of the model. A change in units leads to a rescaling of the quantities. Thus, the independence of the model from the system of units corresponds to a scaling invariance of the model. In cases when the zero point of a unit is arbitrary, we also obtain a translational invariance, but we will not consider translational invariances here.

Suppose that a model involves quantities $(a_1, a_2, \ldots, a_n)$, which may include dependent and independent variables as well as parameters. We denote the dimension of a quantity a by $[a]$. A fundamental system of units is a minimal set of independent units, which we denote symbolically by $(d_1, d_2, \ldots, d_r)$. Different fundamental system of units can be used, but given a fundamental system of units any other derived unit may be constructed uniquely as a product of powers of the fundamental units, so that

$$[a] = d_1^{\alpha_1}d_2^{\alpha_2}\ldots d_r^{\alpha_r} \tag{1.13}$$

for suitable exponents $(\alpha_1, \alpha_2, \ldots, \alpha_r)$.

Example 1.8 In mechanical problems, a fundamental set of units is d_1 = mass, d_2 = length, d_3 = time, or $d_1 = M$, $d_2 = L$, $d_3 = T$, for short. Then velocity

$V = L/T$ and momentum $P = ML/T$ are derived units. We could use instead momentum P, velocity V, and time T as a fundamental system of units, when mass $M = P/V$ and length $L = VT$ are derived units. In problems involving heat flow, we may introduce temperature (measured, for example, in degrees Kelvin) as another fundamental unit, and in problems involving electromagnetism, we may introduce current (measured, for example, in Ampères) as another fundamental unit.

The invariance of a model under the change in units $d_j \mapsto \lambda_j d_j$ implies that it is invariant under the scaling transformation

$$a_i \to \lambda_1^{\alpha_{1,i}} \lambda_2^{\alpha_{2,i}} \dots \lambda_r^{\alpha_{r,i}} a_i \qquad i = 1, \dots, n$$

for any $\lambda_1, \dots \lambda_r > 0$, where

$$[a_i] = d_1^{\alpha_{1,i}} d_2^{\alpha_{2,i}} \dots d_r^{\alpha_{r,i}}. \tag{1.14}$$

Thus, if

$$a = f(a_1, \dots, a_n)$$

is any relation between quantities in the model with the dimensions in (1.13) and (1.14), then f has the scaling property that

$$\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \dots \lambda_r^{\alpha_r} f(a_1, \dots, a_n) = f\left(\lambda_1^{\alpha_{1,1}} \lambda_2^{\alpha_{2,1}} \dots \lambda_r^{\alpha_{r,1}} a_1, \dots, \lambda_1^{\alpha_{1,n}} \lambda_2^{\alpha_{2,n}} \dots \lambda_r^{\alpha_{r,n}} a_n\right).$$

A particular consequence of the invariance of a model under a change of units is that any two quantities which are equal must have the same dimensions. This fact is often useful in finding the dimension of some quantity.

Example 1.9 According to Newton's second law,

$$\text{force} = \text{rate of change of momentum with respect to time.}$$

Thus, if F denotes the dimension of force and P the dimension of momentum, then $F = P/T$. Since $P = MV = ML/T$, we conclude that $F = ML/T^2$ (or mass $\times$ acceleration).

Example 1.10 In fluid mechanics, the shear viscosity μ of a Newtonian fluid is the constant of proportionality that relates the viscous stress tensor $\mathcal{T}$ to the velocity gradient $\nabla \mathbf{u}$:

$$\mathcal{T} = \frac{1}{2}\mu\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right).$$

Stress has dimensions of force/area, so

$$[\mathcal{T}] = \frac{ML}{T^2}\frac{1}{L^2} = \frac{M}{LT^2}.$$

The velocity gradient $\nabla \mathbf{u}$ has dimensions of velocity/length, so

$$[\nabla \mathbf{u}] = \frac{L}{T}\frac{1}{L} = \frac{1}{T}.$$

Equating dimensions, we find that

$$[\mu] = \frac{M}{LT}.$$

We can also write $[\mu] = (M/L^3)(L^2/T)$. It follows that if ρ_0 is the density of the fluid, and $\mu = \rho_0 \nu$, then

$$[\nu] = \frac{L^2}{T}.$$

Thus ν, which is called the kinematical viscosity, has the dimensions of diffusivity. Physically it is the diffusivity of momentum. For example, in time T, viscous effects lead to the diffusion of momentum over a length scale of the order $\sqrt{\nu T}$.

At 20° C, the kinematic viscosity of water is approximately $1\,\mathrm{mm}^2/\mathrm{s}$. Thus, in one second, viscous effects diffuse the fluid momentum over a distance of the order 1 mm.

Scaling invariance implies that we can reduce the number of quantities appearing in the problem by the introduction of dimensionless variables. Suppose that $(a_1, \ldots, a_r)$ are a set of (nonzero) quantities whose dimensions form a fundamental system of units. We denote the remaining quantities in the model by $(b_1, \ldots, b_m)$, where $r + m = n$. Then for suitable exponents $(\beta_{1,i}, \ldots, \beta_{r,i})$, the quantity

$$\Pi_i = \frac{b_i}{a_1^{\beta_{1,i}} \ldots a_r^{\beta_{r,i}}}$$

is dimensionless, meaning that it is invariant under the scaling transformations induced by changes in units. Such dimensionless quantities can often be interpreted as the ratio of two quantities of the same dimension appearing in the problem (such as a ratio of lengths, times, diffusivities, and so on). Perturbation methods are typically applicable when one or more of these dimensionless quantities is small or large.

Any relationship of the form

$$b = f(a_1, \ldots, a_r, b_1, \ldots, b_m)$$

is equivalent to a relation

$$\Pi = f(1, \ldots, 1, \Pi_1, \ldots, \Pi_m).$$

Thus, the introduction of dimensionless quantities reduces the number of variables in the problem by the number of fundamental units involved in the problem. In many cases, nondimensionalization leads to a reduction in the number of parameters in the problem to a minimal number of dimensionless parameters. In some cases,

one may be able to use dimensional arguments to obtain the form of self-similar solutions.

Example 1.11 Consider the following IVP for the Green's function of the heat equation in $\mathbb{R}^d$:

$$\begin{aligned} u_t &= \nu \Delta u, \\ u(x,0) &= E\delta(x). \end{aligned}$$

Here δ is the delta-function. The dimensioned parameters in this problem are the diffusivity ν and the energy E of the point source. The only length and times scales are those that come from the independent variables (x,t), so the solution is self-similar.

We have $[u] = \theta$, where θ denotes temperature, and, since

$$\int_{\mathbb{R}^d} u(x,0)\,dx = E,$$

we have $[E] = \theta L^d$. Dimensional analysis and the rotational invariance of the Laplacian Δ imply that

$$u(x,t) = \frac{E}{(\nu t)^{d/2}} f\left(\frac{|x|}{\sqrt{\nu t}}\right).$$

Using this expression for $u(x,t)$ in the PDE, we get an ODE for $f(\xi)$,

$$f'' + \left(\frac{\xi}{2} + \frac{d-1}{\xi}\right) f' + \frac{d}{2} f = 0.$$

We can rewrite this equation as a first-order ODE for $f' + \frac{\xi}{2} f$,

$$\left(f' + \frac{\xi}{2} f\right)' + \frac{d-1}{\xi}\left(f' + \frac{\xi}{2} f\right) = 0.$$

Solving this equation, we get

$$f' + \frac{\xi}{2} f = \frac{b}{\xi^{d-1}},$$

where b is a constant of integration. Solving for f, we get

$$f(\xi) = a e^{-\xi^2/4} + b e^{-\xi^2/4} \int \frac{e^{-\xi^2}}{\xi^{d-1}}\,d\xi,$$

where a s another constant of integration. In order for f to be integrable, we must set $b = 0$. Then

$$u(x,t) = \frac{aE}{(\nu t)^{d/2}} \exp\left(-\frac{|x|^2}{4\nu t}\right).$$

Imposing the requirement that

$$\int_{\mathbb{R}^d} u(x,t)\,dx = E,$$

and using the standard integral

$$\int_{\mathbb{R}^d} \exp\left(-\frac{|x|^2}{2c}\right)\,dx = (2\pi c)^{d/2},$$

we find that $a = (4\pi)^{-d/2}$, and

$$u(x,t) = \frac{E}{(4\pi\nu t)^{d/2}} \exp\left(-\frac{|x|^2}{4\nu t}\right).$$

Example 1.12 Consider a sphere of radius L moving through a fluid with constant speed U. A primary quantity of interest is the total drag force D exerted by the fluid on the sphere. We assume that the fluid is incompressible, which is a good approximation if the flow speed U is much less than the speed of sound in the fluid. The fluid properties are then determined by the density ρ_0 and the kinematic viscosity ν. Hence,

$$D = f(U, L, \rho_0, \nu).$$

Since the drag D has the dimensions of force (ML/T^2), dimensional analysis implies that

$$D = \rho_0 U^2 L^2 F\left(\frac{UL}{\nu}\right).$$

Thus, the dimensionless drag

$$\frac{D}{\rho_0 U^2 L^2} = F(\mathrm{Re})$$

is a function of the Reynold's number

$$\mathrm{Re} = \frac{UL}{\nu}.$$

The function F has a complicated dependence on Re that is difficult to compute explicitly. For example, F changes rapidly near Reynolds numbers for which the flow past the sphere becomes turbulent. Nevertheless, experimental measurements agree very well with the result of this dimensionless analysis (see Figure 1.9 in [1], for example).

The equations of motion of the fluid are the incompressible Navier-Stokes equations,

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla p = \nu \Delta u, \\ &\nabla \cdot u = 0. \end{aligned}$$

To nondimensionalize these equations with respect to (U, L, ρ), we introduce dimensionless variables

$$u^* = \frac{u}{U}, \quad p^* = \frac{p}{\rho U^2}, \quad x^* = \frac{x}{L}, \quad t^* = \frac{Ut}{L},$$

and find that

$$\begin{aligned} &u^*_{t^*} + u^* \cdot \nabla^* u^* + \nabla^* p^* = \varepsilon \Delta^* u^*, \\ &\nabla^* \cdot u^* = 0. \end{aligned}$$

Here,

$$\varepsilon = \frac{\nu}{UL} = \frac{1}{\mathrm{Re}}.$$

The boundary conditions correspond to a flow of speed 1 past a sphere of radius 1. Thus, assuming that no other parameters enter into the problem, the drag computed from the solution of these equations depends only on ε, as obtained from the dimensional analysis above.

Dimensional analysis leads to continuous scaling symmetries. These scaling symmetries are not the only continuous symmetries possessed by differential equations. The theory of Lie groups and Lie algebras provides a systematic method for computing all continuous symmetries of a given differential equation [18]. Lie originally introduced the notions of Lie groups and Lie algebras precisely for this purpose.

Example 1.13 The full group of symmetries of the one-dimensional heat equation

$$u_t = u_{xx}$$

is generated by the following transformations [18]:

$$\begin{aligned} &u(x,t) \mapsto u(x-\alpha, t), \\ &u(x,t) \mapsto u(x, t-\beta), \\ &u(x,t) \mapsto \gamma u(x,t), \\ &u(x,t) \mapsto u(\delta x, \delta^2 t), \\ &u(x,t) \mapsto e^{-\varepsilon x + \varepsilon^2 t} u(x - 2\varepsilon t, t), \\ &u(x,t) \mapsto \frac{1}{\sqrt{1+4\eta t}} \exp\left[\frac{-\eta x^2}{1+4\eta t}\right] u\left(\frac{x}{1+4\eta t}, \frac{t}{1+4\eta t}\right), \\ &u(x,t) \mapsto u(x,t) + v(x,t), \end{aligned}$$

where $(\alpha, \ldots, \eta)$ are constants, and $v(x,t)$ is an arbitrary solution of the heat equation. The scaling symmetries involving γ and δ can be deduced by dimensional arguments, but the symmetries involving ε and η cannot.

For further discussion of dimensional analysis and self-similar solutions, see [1].

Chapter 2
Asymptotic Expansions

In this chapter, we define the order notation and asymptotic expansions. For additional discussion, see [4], and [17].

2.1 Order notation

The O and o order notation provides a precise mathematical formulation of ideas that correspond — roughly — to the 'same order of magnitude' and 'smaller order of magnitude.' We state the definitions for the asymptotic behavior of a real-valued function $f(x)$ as $x \to 0$, where x is a real parameter. With obvious modifications, similar definitions apply to asymptotic behavior in the limits $x \to 0^+$, $x \to x_0$, $x \to \infty$, to complex or integer parameters, and other cases. Also, by replacing $|\cdot|$ with a norm, we can define similiar concepts for functions taking values in a normed linear space.

Definition 2.1 Let $f, g : \mathbb{R} \setminus 0 \to \mathbb{R}$ be real functions. We say $f = O(g)$ as $x \to 0$ if there are constants C and $r > 0$ such that

$$|f(x)| \leq C|g(x)| \text{ whenever } 0 < |x| < r.$$

We say $f = o(g)$ as $x \to 0$ if for every $\delta > 0$ there is an $r > 0$ such that

$$|f(x)| \leq \delta|g(x)| \text{ whenever } 0 < |x| < r.$$

If $g \neq 0$, then $f = O(g)$ as $x \to 0$ if and only if f/g is bounded in a (punctured) neighborhood of 0, and $f = o(g)$ if and only if $f/g \to 0$ as $x \to 0$.

We also write $f \ll g$, or f is 'much less than' g, if $f = o(g)$, and $f \sim g$, or f is asymptotic to g, if $f/g \to 1$.

Example 2.2 A few simple examples are:

(a) $\sin 1/x = O(1)$ as $x \to 0$

(b) it is not true that $1 = O(\sin 1/x)$ as $x \to 0$, because $\sin 1/x$ vanishes is every neighborhood of $x = 0$;
(c) $x^3 = o(x^2)$ as $x \to 0$, and $x^2 = o(x^3)$ as $x \to \infty$;
(d) $x = o(\log x)$ as $x \to 0^+$, and $\log x = o(x)$ as $x \to \infty$;
(e) $\sin x \sim x$ as $x \to 0$;
(f) $e^{-1/x} = o(x^n)$ as $x \to 0^+$ for any $n \in \mathbb{N}$.

The o and O notations are not quantitative without estimates for the constants C, δ, and r appearing in the definitions.

2.2 Asymptotic expansions

An asymptotic expansion describes the asymptotic behavior of a function in terms of a sequence of gauge functions. The definition was introduced by Poincaré (1886), and it provides a solid mathematical foundation for the use of many divergent series.

Definition 2.3 A sequence of functions $\varphi_n : \mathbb{R} \setminus 0 \to \mathbb{R}$, where $n = 0, 1, 2, \ldots$, is an *asymptotic sequence* as $x \to 0$ if for each $n = 0, 1, 2, \ldots$ we have

$$\varphi_{n+1} = o(\varphi_n) \qquad \text{as } x \to 0.$$

We call φ_n a *gauge function*. If $\{\varphi_n\}$ is an asymptotic sequence and $f : \mathbb{R} \setminus 0 \to \mathbb{R}$ is a function, we write

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x) \qquad \text{as } x \to 0 \tag{2.1}$$

if for each $N = 0, 1, 2, \ldots$ we have

$$f(x) - \sum_{n=0}^{N} a_n \varphi_n(x) = o(\varphi_N) \qquad \text{as } x \to 0.$$

We call (2.1) the asymptotic expansion of f with respect to $\{\varphi_n\}$ as $x \to 0$.

Example 2.4 The functions $\varphi_n(x) = x^n$ form an asymptotic sequence as $x \to 0^+$. Asymptotic expansions with respect to this sequence are called asymptotic power series, and they are discussed further below. The functions $\varphi_n(x) = x^{-n}$ form an asymptotic sequence as $x \to \infty$.

Example 2.5 The function $\log \sin x$ has an asymptotic expansion as $x \to 0^+$ with respect to the asymptotic sequence $\{\log x, x^2, x^4, \ldots\}$:

$$\log \sin x \sim \log x + \frac{1}{6}x^2 + \ldots \qquad \text{as } x \to 0^+.$$

If, as is usually the case, the gauge functions φ_n do not vanish in a punctured neighborhood of 0, then it follows from Definition 2.1 that

$$a_{N+1} = \lim_{x \to 0} \frac{f(x) - \sum_{n=0}^{N} a_n \varphi_n(x)}{\varphi_{N+1}}.$$

Thus, if a function has an expansion with respect to a given sequence of gauge functions, the expansion is unique. Different functions may have the same asymptotic expansion.

Example 2.6 For any constant $c \in \mathbb{R}$, we have

$$\frac{1}{1-x} + ce^{-1/x} \sim 1 + x + x^2 + \ldots + x^n + \ldots \qquad \text{as } x \to 0^+,$$

since $e^{-1/x} = o(x^n)$ as $x \to 0^+$ for every $n \in \mathbb{N}$.

Asymptotic expansions can be added, and — under natural conditions on the gauge functions — multiplied. The term-by-term integration of asymptotic expansions is valid, but differentiation may not be, because small, highly-oscillatory terms can become large when they are differentiated.

Example 2.7 Let

$$f(x) = \frac{1}{1-x} + e^{-1/x} \sin e^{1/x}.$$

Then

$$f(x) \sim 1 + x + x^2 + x^3 + \ldots \qquad \text{as } x \to 0^+,$$

but

$$f'(x) \sim -\frac{\cos e^{1/x}}{x^2} + 1 + 2x + 3x^2 + \ldots \qquad \text{as } x \to 0^+.$$

Term-by-term differentiation is valid under suitable assumptions that rule out the presence of small, highly oscillatory terms. For example, a convergent power series expansion of an analytic function can be differentiated term-by-term

2.2.1 *Asymptotic power series*

Asymptotic power series,

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \qquad \text{as } x \to 0,$$

are among the most common and useful asymptotic expansions.

If f is a smooth (C^∞) function in a neighborhood of the origin, then Taylor's theorem implies that

$$\left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \right| \leq C_{N+1} x^{N+1} \qquad \text{when } |x| \leq r,$$

where

$$C_{N+1} = \sup_{|x| \leq r} \frac{\left| f^{(N+1)}(x) \right|}{(N+1)!}.$$

It follows that f has the asymptotic power series expansion

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad \text{as } x \to 0. \tag{2.2}$$

The asymptotic power series in (2.2) converges to f in a neighborhood of the origin if and only if f is analytic at $x = 0$. If f is C^∞ but not analytic, the series may converge to a function different from f (see Example 2.6 with $c \neq 0$) or it may diverge (see (2.4) or (3.3) below).

The Taylor series of $f(x)$ at $x = x_0$,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n,$$

does not provide an asymptotic expansion of $f(x)$ as $x \to x_1$ when $x_1 \neq x_0$ even if it converges. The partial sums therefore do not generally provide a good approximation of $f(x)$ as $x \to x_1$. (See Example 2.9, where a partial sum of the Taylor series of the error function at $x = 0$ provides a poor approximation of the function when x is large.)

The following (rather surprising) theorem shows that there are no restrictions on the growth rate of the coefficients in an asymptotic power series, unlike the case of convergent power series.

Theorem 2.8 (Borel-Ritt) Given any sequence $\{a_n\}$ of real (or complex) coefficients, there exists a C^∞-function $f : \mathbb{R} \to \mathbb{R}$ (or $f : \mathbb{R} \to \mathbb{C}$) such that

$$f(x) \sim \sum a_n x^n \qquad \text{as } x \to 0.$$

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that

$$\eta(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

We choose a sequence of positive numbers $\{\delta_n\}$ such that $\delta_n \to 0$ as $n \to \infty$ and

$$|a_n| \left\| x^n \eta\left(\frac{x}{\delta_n}\right) \right\|_{C^n} \leq \frac{1}{2^n}, \tag{2.3}$$

where

$$\|f\|_{C^n} = \sup_{x\in\mathbb{R}} \sum_{k=0}^{n} \left| f^{(k)}(x) \right|$$

denotes the C^n-norm. We define

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \eta\left(\frac{x}{\delta_n}\right).$$

This series converges pointwise, since when $x = 0$ it is equal to a_0, and when $x \neq 0$ it consists of only finitely many terms. The condition in (2.3) implies that the sequence converges in C^n for every $n \in \mathbb{N}$. Hence, the function f has continuous derivatives of all orders. □

2.2.2 *Asymptotic versus convergent series*

We have already observed that an asymptotic series need not be convergent, and a convergent series need not be asymptotic. To explain the difference in more detail, we consider a formal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x),$$

where $\{a_n\}$ is a sequence of coefficients and $\{\varphi_n(x)\}$ is an asymptotic sequence as $x \to 0$. We denote the partial sums by

$$S_N(x) = \sum_{n=0}^{N} a_n \varphi_n(x).$$

Then convergence is concerned with the behavior of $S_N(x)$ as $N \to \infty$ with x fixed, whereas asymptoticity (at $x = 0$) is concerned with the behavior of $S_N(x)$ as $x \to 0$ with N fixed.

A convergent series define a unique limiting sum, but convergence does not give any indication of how rapidly the series it converges, nor of how well the sum of a fixed number of terms approximates the limit. An asymptotic series does not define a unique sum, nor does it provide an arbitrarily accurate approximation of the value of a function it represents at any $x \neq 0$, but its partial sums provide good approximations of these functions that when x is sufficiently small.

The following example illustrates the contrast between convergent and asymptotic series. We will examine another example of a divergent asymptotic series in Section 3.1.

Example 2.9 The error function $\operatorname{erf} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Integrating the power series expansion of e^{-t^2} term by term, we obtain the power series expansion of $\operatorname{erf} x$,

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \left\{ x - \frac{1}{3}x^3 + \ldots + \frac{(-1)^n}{(2n+1)n!}x^{2n+1} + \ldots \right\},$$

which is convergent for every $x \in \mathbb{R}$. For large values of x, however, the convergence is very slow. the Taylor series of the error function at $x = 0$. Instead, we can use the following divergent asymptotic expansion, proved below, to obtain accurate approximations of $\operatorname{erf} x$ for large x:

$$\operatorname{erf} x \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(2n-1)!!}{2^n} \frac{1}{x^{n+1}} \qquad \text{as } x \to \infty, \tag{2.4}$$

where $(2n-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$. For example, when $x = 3$, we need 31 terms in the Taylor series at $x = 0$ to approximate $\operatorname{erf} 3$ to an accuracy of 10^{-5}, whereas we only need 2 terms in the asymptotic expansion.

Proposition 2.10 The expansion (2.4) is an asymptotic expansion of $\operatorname{erf} x$.

Proof. We write

$$\operatorname{erf} x = 1 - \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt,$$

and make the change of variables $s = t^2$,

$$\operatorname{erf} x = 1 - \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} s^{-1/2} e^{-s}\, ds.$$

For $n = 0, 1, 2, \ldots$, we define

$$F_n(x) = \int_{x^2}^{\infty} s^{-n-1/2} e^{-s}\, ds.$$

Then an integration by parts implies that

$$F_n(x) = \frac{e^{-x^2}}{x^{2n+1}} - \left(n + \frac{1}{2}\right) F_{n+1}(x).$$

By repeated use of this recursion relation, we find that

$$\begin{aligned}
\operatorname{erf} x &= 1 - \frac{1}{\sqrt{\pi}} F_0(x) \\
&= 1 - \frac{1}{\sqrt{\pi}} \left[\frac{e^{-x^2}}{x} - \frac{1}{2} F_1(x) \right] \\
&= 1 - \frac{1}{\sqrt{\pi}} \left[e^{-x^2} \left(\frac{1}{x} - \frac{1}{2x^3} \right) + \frac{1 \cdot 3}{2^2} F_2(x) \right]
\end{aligned}$$

$$= 1 - \frac{1}{\sqrt{\pi}} \left[e^{-x^2} \left(\frac{1}{x} - \frac{1}{2x^3} + \ldots (-1)^N \frac{1 \cdot 3 \cdot \ldots \cdot (2N-1)}{2^N x^{2N+1}} \right) + (-1)^{N+1} \frac{1 \cdot 3 \cdot \ldots \cdot (2N+1)}{2^{N+1}} F_{N+1}(x) \right].$$

It follows that

$$\operatorname{erf} x = 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{N} (-1)^n \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{2^n x^{2n+1}} + R_{N+1}(x)$$

where

$$R_{N+1}(x) = (-1)^{N+1} \frac{1}{\sqrt{\pi}} \frac{1 \cdot 3 \cdot \ldots \cdot (2N+1)}{2^{N+1}} F_{N+1}(x).$$

Since

$$\begin{aligned} |F_n(x)| &= \left| \int_{x^2}^{\infty} s^{-n-1/2} e^{-s} \, ds \right| \\ &\leq \frac{1}{x^{2n+1}} \int_{x^2}^{\infty} e^{-s} \, ds \\ &\leq \frac{e^{-x^2}}{x^{2n+1}}, \end{aligned}$$

we have

$$|R_{N+1}(x)| \leq C_{N+1} \frac{e^{-x^2}}{x^{2N+3}},$$

where

$$C_N = \frac{1 \cdot 3 \cdot \ldots \cdot (2N+1)}{2^{N+1} \sqrt{\pi}}.$$

This proves the result. □

2.2.3 *Generalized asymptotic expansions*

Sometimes it is useful to consider more general asymptotic expansions with respect to a sequence of gauge functions $\{\varphi_n\}$ of the form

$$f(x) \sim \sum_{n=0}^{\infty} f_n(x),$$

where for each $N = 0, 1, 2, \ldots$

$$f(x) - \sum_{n=0}^{N} f_n(x) = o(\varphi_{N+1}).$$

For example, an expansion of the form

$$f(x) \sim \sum_{n=0}^{\infty} a_n(x)\varphi_n(x)$$

in which the coefficients a_n are bounded functions of x is a generalized asymptotic expansion. Such expansions provide additional flexibility, but they are not unique and have to be used with care in many cases.

Example 2.11 We have

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n \qquad \text{as } x \to 0^+.$$

We also have

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} (1+x)x^{2n} \qquad \text{as } x \to 0^+.$$

This is a generalized asymptotic expansion with respect to $\{x^n \mid n = 0, 1, 2, \ldots\}$ that differs from the first one.

Example 2.12 According to [17], the following generalized asymptotic expansion with respect to the asymptotic sequence $\{(\log x)^{-n}\}$

$$\frac{\sin x}{x} \sim \sum_{n=1}^{\infty} \frac{n! e^{-(n+1)x/(2n)}}{(\log x)^n} \qquad \text{as } x \to \infty$$

is an example showing that "the definition admits expansions that have no conceivable value."

Example 2.13 The method of matched asymptotic expansions and the method of multiple scales lead to generalized asymptotic expansions, in which the generalized expansions have a definite form dictated by the corresponding methods.

Example 2.14 A physical example of a generalized asymptotic expansion arises in the derivation of the Navier-Stokes equations of fluid mechanics from the Boltzmann equations of kinetic theory by means of the Chapman-Enskog expansion. If λ is the mean free path of the fluid molecules and L is a macroscopic length-scale of the fluid flow, then the relevant small parameter is

$$\varepsilon = \frac{\lambda}{L} \ll 1.$$

The leading-order term in the Chapman-Enskog expansion satisfies the Navier-Stokes equations in which the fluid viscosity is of the order ε when nondimensionalized by the length and time scales characteristic of the fluid flow. Thus the leading order solution depends on the perturbation parameter ε, and this expansion is a generalized asymptotic expansion.

2.2.4 *Nonuniform asymptotic expansions*

In many problems, we seek an asymptotic expansion as $\varepsilon \to 0$ of a function $u(x, \varepsilon)$, where x is an independent variable.* The asymptotic behavior of the function with respect to ε may depend upon x, in which case we say that the expansion is nonuniform.

Example 2.15 Consider the function $u : [0, \infty) \times (0, \infty) \to \mathbb{R}$ defined by

$$u(x, \varepsilon) = \frac{1}{x + \varepsilon}.$$

If $x > 0$, then

$$u(x, \varepsilon) \sim \frac{1}{x} \left[1 - \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} + \ldots \right] \qquad \text{as } \varepsilon \to 0^+.$$

On the other hand, if $x = 0$, then

$$u(0, \varepsilon) \sim \frac{1}{\varepsilon} \qquad \text{as } \varepsilon \to 0^+.$$

The transition between these two different expansions occurs when $x = O(\varepsilon)$. In the limit $\varepsilon \to 0^+$ with $y = x/\varepsilon$ fixed, we have

$$u(\varepsilon y, \varepsilon) \sim \frac{1}{\varepsilon} \left(\frac{1}{y + 1} \right) \qquad \text{as } \varepsilon \to 0^+.$$

This expansion matches with the other two expansions in the limits $y \to \infty$ and $y \to 0^+$.

Nonuniform asymptotic expansions are not a mathematical pathology, and they are often the crucial feature of singular perturbation problems. We will encounter many such problems below; for example, in boundary layer problems the solution has different asymptotic expansions inside and outside the boundary layer, and in various problems involving oscillators nonuniformities aries for long times.

2.3 Stokes phenomenon

An asymptotic expansion as $z \to \infty$ of a complex function $f : \mathbb{C} \to \mathbb{C}$ with an essential singularity at $z = \infty$ is typically valid only in a wedge-shaped region $\alpha < \arg z < \beta$, and the function has different asymptotic expansions in different wedges.† The change in the form of the asymptotic expansion across the boundaries of the wedges is called the Stokes phenomenon.

*We consider asymptotic expansions with respect to ε, not x.

†We consider a function with an essential singularity at $z = \infty$ for definiteness; the same phenomenon occurs for functions with an essential singularity at any $z_0 \in \mathbb{C}$.

Example 2.16 Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \sinh z^2 = \frac{e^{z^2} - e^{-z^2}}{2}.$$

Let $|z| \to \infty$ with $\arg z$ fixed, and define

$$\Omega_1 = \{z \in \mathbb{C} \mid -\pi/4 < \arg z < \pi/4\},$$
$$\Omega_2 = \{z \in \mathbb{C} \mid \pi/4 < \arg z < 3\pi/4 \text{ or } -3\pi/4 < \arg z < -\pi/4\}.$$

If $z \in \Omega_1$, then $\operatorname{Re} z^2 > 0$ and $e^{z^2} \gg e^{-z^2}$, whereas if $z \in \Omega_2$, then $\operatorname{Re} z^2 < 0$ and $e^{z^2} \ll e^{-z^2}$. Hence

$$f(z) \sim \begin{cases} \frac{1}{2}e^{z^2} & \text{as } |z| \to \infty \text{ in } \Omega_1, \\ \frac{1}{2}e^{-z^2} & \text{as } |z| \to \infty \text{ in } \Omega_2. \end{cases}$$

The lines $\arg z = \pm\pi/4, \pm 3\pi/4$ where the asymptotic expansion changes form are called anti-Stokes lines. The terms e^{z^2} and e^{-z^2} switch from being dominant to subdominant as z crosses an anti-Stokes lines. The lines $\arg z = 0, \pi, \pm\pi/2$ where the subdominant term is as small as possible relative to the dominant term are called Stokes lines.

This example concerns a simple explicit function, but a similar behavior occurs for solutions of ODEs with essential singularities in the complex plane, such as error functions, Airy functions, and Bessel functions.

Example 2.17 The error function can be extended to an entire function $\operatorname{erf} : \mathbb{C} \to \mathbb{C}$ with an essential singularity at $z = \infty$. It has the following asymptotic expansions in different wedges:

$$\operatorname{erf} z \sim \begin{cases} 1 - \exp(-z^2)/(z\sqrt{\pi}) & \text{as } z \to \infty \text{ with } z \in \Omega_1, \\ -1 - \exp(-z^2)/(z\sqrt{\pi}) & \text{as } z \to \infty \text{ with } z \in \Omega_2, \\ -\exp(-z^2)/(z\sqrt{\pi}) & \text{as } z \to \infty \text{ with } z \in \Omega_3. \end{cases}$$

where

$$\Omega_1 = \{z \in \mathbb{C} \mid -\pi/4 < \arg z < \pi/4\},$$
$$\Omega_2 = \{z \in \mathbb{C} \mid 3\pi/4 < \arg z < 5\pi/4\},$$
$$\Omega_3 = \{z \in \mathbb{C} \mid \pi/4 < \arg z < 3\pi/4 \text{ or } 5\pi/4 < \arg z < 7\pi/4\}.$$

Often one wants to determine the asymptotic behavior of such a function in one wedge given its behavior in another wedge. This is called a connection problem (see Section 3.4 for the case of the Airy function). The apparently discontinuous change in the form of the asymptotic expansion of the solutions of an ODE across an anti-Stokes line can be understood using exponential asymptotics as the result of a continuous, but rapid, change in the coefficient of the subdominant terms across the Stokes line [2].

Chapter 3

Asymptotic Expansion of Integrals

In this chapter, we give some examples of asymptotic expansions of integrals. We do not attempt to give a complete discussion of this subject (see [4], [21] for more information).

3.1 Euler's integral

Consider the following integral (Euler, 1754):

$$I(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\, dt, \tag{3.1}$$

where $x \geq 0$.

First, we proceed formally. We use the power series expansion

$$\frac{1}{1+xt} = 1 - xt + x^2t^2 + \ldots + (-1)^n x^n t^n + \ldots \tag{3.2}$$

inside the integral in (3.1), and integrate the result term-by-term. Using the integral

$$\int_0^\infty t^n e^{-t}\, dx = n!,$$

we get

$$I(x) \sim 1 - x + 2!x^2 + \ldots + (-1)^n n! x^n + \ldots. \tag{3.3}$$

The coefficients in this power series grow factorially, and the terms diverge as $n \to \infty$. Thus, the series does not converge for any $x \neq 0$. On the other hand, the following proposition shows that the series is an asymptotic expansion of $I(x)$ as $x \to 0^+$, and the the error between a partial sum and the integral is less than the first term neglected in the asymptotic series. The proof also illustrates the use of integration by parts in deriving an asymptotic expansion.

Proposition 3.1 For $x \geq 0$ and $N = 0, 1, 2, \ldots$, we have

$$\left| I(x) - \left\{ 1 - x + \ldots + (-1)^N N! x^N \right\} \right| \leq (N+1)!\, x^{N+1}.$$

Proof. Integrating by parts in (3.1), we have

$$I(x) = 1 - x \int_0^\infty \frac{e^{-t}}{(1+xt)^2}\, dt.$$

After $N+1$ integrations by parts, we find that

$$I(x) = 1 - x + \ldots + (-1)^N N! x^N + R_{N+1}(x),$$

where

$$R_{N+1}(x) = (-1)^{N+1}(N+1)! x^{N+1} \int_0^\infty \frac{e^{-t}}{(1+xt)^{N+2}}\, dt.$$

Estimating R_{N+1} for $x \geq 0$, we find that

$$\begin{aligned} |R_{N+1}(x)| &\leq (N+1)! x^{N+1} \int_0^\infty e^{-t}\, dt \\ &\leq (N+1)!\, x^{N+1} \end{aligned}$$

which proves the result. Equation (3.1) shows that the partial sums oscillate above (N even) and below (N odd) the value of the integral. □

Heuristically, the lack of convergence of the series in (3.3) is a consequence of the fact that the power series expansion (3.2) does not converge over the whole integration region, but only when $0 \leq t < 1/x$. On the other hand, when x is small, the expansion is convergent over most of the integration region, and the integrand is exponentially small when it is not. This explains the accuracy of the resulting partial sum approximations.

The integral in (3.1) is not well-defined when $x < 0$ since then the integrand has a nonintegrable singularity at $t = -1/x$. The fact that $x = 0$ is a 'transition point' is associated with the lack of convergence of the asymptotic power series, because any convergent power series would converge in a disk (in $\mathbb{C}$) centered at $x = 0$.

Since the asymptotic series is not convergent, its partial sums do not provide an arbitrarily accurate approximation of $I(x)$ for a fixed $x > 0$. It is interesting to ask, however, what partial sum gives the the best approximation.

This occurs when n minimizes the remainder $R_{n+1}(x)$. The remainder decreases when $n \leq x$ and increases when $n + 1 > x$, so the best approximation occurs when $n + 1 \approx [1/x]$, and then $R_{n+1}(x) \approx (1/x)! x^{1/x}$. Using Stirling's formula (see Example 3.10),

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n} \qquad \text{as } n \to \infty,$$

we find that the optimal truncation at $n \approx [1/x] - 1$ gives an error

$$R_n(x) \sim \sqrt{\frac{2\pi}{x}}\, e^{-1/x} \qquad \text{as } x \to 0^+.$$

Thus, even though each partial sum with a fixed number of terms is polynomially accurate in x, the optimal partial sum approximation is exponentially accurate.

Example 3.2 The partial sums

$$S_N(x) = 1 - x + \ldots + (-1)^N N! x^N$$

for $x = 0.1$ and $2 \leq N \leq 15$ are given in the following table (to an appropriate accuracy).

N	$S_N(0.1)$
2	0.9
3	0.92
4	0.914
5	0.9164
6	0.9152
7	0.91529
8	0.915416
9	0.915819
10	0.915463
11	0.915819
12	0.91542
13	0.9159
14	0.9153
15	0.9162

It follows that

$$0.91546 \leq I(0.1) \leq 0.91582.$$

Numerical integration shows that, to four significant figures,

$$I(0.1) \approx 0.9156.$$

In some problems, the exponentially small corrections to a power series asymptotic expansion contain important information. For example, the vanishing or nonvanishing of these corrections may correspond to the existence or nonexistence of particular types of solutions of PDEs, such as traveling waves or breathers. There exist methods of exponential asymptotics, or asymptotics beyond all orders, that can be used to compute exponentially small terms.

3.2 Perturbed Gaussian integrals

Consider the following integral

$$I(a,\varepsilon) = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}ax^2 - \varepsilon x^4\right] dx, \tag{3.4}$$

where $a > 0$ and $\varepsilon \geq 0$. For $\varepsilon = 0$, this is a standard Gaussian integral, and

$$I(a,0) = \frac{1}{\sqrt{2\pi a}}.$$

For $\varepsilon > 0$, we cannot compute $I(a,\varepsilon)$ explicitly, but we can obtain an asymptotic expansion as $\varepsilon \to 0^+$.

First, we proceed formally. Taylor expanding the exponential with respect to ε,

$$\exp\left[-\frac{1}{2}ax^2 - \varepsilon x^4\right] = e^{-\frac{1}{2}ax^2}\left\{1 - \varepsilon x^4 + \frac{1}{2!}\varepsilon^2 x^8 + \ldots + \frac{(-1)^n}{n!}\varepsilon^n x^{4n} + \ldots\right\},$$

and integrating the result term-by-term, we get

$$I(a,\varepsilon) \sim \frac{1}{\sqrt{2\pi a}}\left\{1 - \varepsilon\langle x^4\rangle + \ldots + \frac{(-1)^n}{n!}\varepsilon^n\langle x^{4n}\rangle + \ldots\right\},, \tag{3.5}$$

where

$$\langle x^{4n}\rangle = \frac{\int_{-\infty}^{\infty} x^{4n} e^{-\frac{1}{2}ax^2}\, dx}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2}\, dx}.$$

We use a special case of Wick's theorem to calculate these integrals.

Proposition 3.3 For $m \in \mathbb{N}$, we have

$$\langle x^{2m}\rangle = \frac{(2m-1)!!}{a^m},$$

where

$$(2m-1)!! = 1 \cdot 3 \cdot 5 \ldots (2m-3) \cdot (2m-1).$$

Proof. Let

$$J(a,b) = \frac{\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2 + bx}\, dx}{\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2}\, dx}.$$

Differentiating $J(a,b)$ n-times with respect to b and setting $b = 0$, we find that

$$\langle x^n\rangle = \left.\frac{d^n}{db^n} J(a,b)\right|_{b=0}.$$

Writing

$$e^{-\frac{1}{2}ax^2 + bx} = e^{-\frac{1}{2}a(x-b)^2 + \frac{b^2}{2a}}$$

and making the change of variable $(x-b) \mapsto x$ in the numerator, we deduce that

$$J(a,b) = e^{\frac{b^2}{2a}}.$$

Hence,

$$\begin{aligned}\langle x^n \rangle &= \frac{d^n}{db^n}\left[e^{\frac{b^2}{2a}}\right]\Big|_{b=0} \\ &= \frac{d^n}{db^n}\left\{1 + \frac{b^2}{2a} + \ldots + \frac{1}{m!}\frac{b^{2m}}{(2a)^m} + \ldots\right\}\Big|_{b=0},\end{aligned}$$

which implies that

$$\langle x^{2m} \rangle = \frac{(2m)!}{(2a)^m m!}.$$

This expression is equivalent to the result. □

Using the result of this proposition in (3.5), we conclude that

$$I(a,\varepsilon) \sim \frac{1}{\sqrt{2\pi a}}\left[1 - \frac{3}{a^2}\varepsilon + \frac{105}{a^4}\varepsilon^2 + \ldots + a_n\varepsilon^n + \ldots\right] \qquad \text{as } \varepsilon \to 0^+, \tag{3.6}$$

where

$$a_n = \frac{(-1)^n(4n-1)!!}{n!a^{2n}}. \tag{3.7}$$

By the ratio test, the radius of convergence R of this series is

$$\begin{aligned}R &= \lim_{n\to\infty} \frac{(n+1)!a^{2n+2}(4n-1)!}{n!a^{2n}(4n+3)!!} \\ &= \lim_{n\to\infty} \frac{(n+1)a^2}{(4n+1)(4n+3)} \\ &= 0.\end{aligned}$$

Thus, the series diverges for every $\varepsilon > 0$, as could have been anticipated by the fact that $I(a,\varepsilon)$ is undefined for $\varepsilon < 0$.

The next proposition shows that the series is an asymptotic expansion of $I(a,\varepsilon)$ as $\varepsilon \to 0^+$.

Proposition 3.4 Suppose $I(a,\varepsilon)$ is defined by (3.4). For each $N = 0, 1, 2, \ldots$ and $\varepsilon > 0$, we have

$$\left| I(a,\varepsilon) - \sum_{n=0}^{N} a_n\varepsilon^n \right| \le C_{N+1}\varepsilon^{N+1}$$

where a_n is given by (3.7), and

$$C_{N+1} = \frac{1}{(N+1)!}\int_{-\infty}^{\infty} x^{4(N+1)}e^{-\frac{1}{2}ax^2}\,dx.$$

Proof. Taylor's theorem implies that for $y \geq 0$ and $N \in \mathbb{N}$

$$e^{-y} = 1 - y + \frac{1}{2!}y^2 + \ldots + \frac{(-1)^N}{N!}y^N + s_{N+1}(y),$$

where

$$s_{N+1}(y) = \frac{1}{(N+1)!}\,\frac{d^{N+1}}{dy^{N+1}}\left(e^{-y}\right)\Big|_{y=\eta} y^{N+1}$$

for some $0 \leq \eta \leq y$. Replacing y by εx^4 in this equation and estimating the remainder, we find that

$$e^{-\varepsilon x^4} = 1 - \varepsilon x^4 + \frac{1}{2!}\varepsilon^2 x^4 + \ldots + \frac{(-1)^N}{N!}\varepsilon^N x^{4N} + \varepsilon^{N+1} r_{N+1}(x), \tag{3.8}$$

where

$$|r_{N+1}(x)| \leq \frac{x^{4(N+1)}}{(N+1)!}.$$

Using (3.8) in (3.4), we get

$$I(a,\varepsilon) = \sum_{n=0}^{N} a_n \varepsilon^n + \varepsilon^{N+1} \int_{-\infty}^{\infty} r_{N+1}(x) e^{-\frac{1}{2}ax^2}\, dx.$$

It follows that

$$\begin{aligned}\left| I(a,\varepsilon) - \sum_{n=0}^{N} a_n \varepsilon^n \right| &\leq \varepsilon^{N+1} \int_{-\infty}^{\infty} |r_{N+1}(x)|\, e^{-\frac{1}{2}ax^2}\, dx \\ &\leq \varepsilon^{N+1} \frac{1}{(N+1)!} \int_{-\infty}^{\infty} x^{4(N+1)} e^{-\frac{1}{2}ax^2}\, dx,\end{aligned}$$

which proves the result. □

These expansions generalize to multi-dimensional Gaussian integrals, of the form

$$I(A,\varepsilon) = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}x^T A x + \varepsilon V(x)\right) dx$$

where A is a symmetric $n \times n$ matrix, and to infinite-dimensional functional integrals, such as those given by the formal expression

$$I(\varepsilon) = \int \exp\left\{-\int \left(\frac{1}{2}|\nabla u(x)|^2 + \frac{1}{2}u^2(x) + \varepsilon V\left(u(x)\right)\right) dx\right\} Du$$

which appear in quantum field theory and statistical physics.

3.3 The method of stationary phase

The method of stationary phase provides an asymptotic expansion of integrals with a rapidly oscillating integrand. Because of cancelation, the behavior of such integrals is dominated by contributions from neighborhoods of the stationary phase points where the oscillations are the slowest.

Example 3.5 Consider the following Fresnel integral

$$I(\varepsilon) = \int_{-\infty}^{\infty} e^{it^2/\varepsilon}\, dt.$$

This oscillatory integral is not defined as an absolutely convergent integral, since $\left|e^{it^2/\varepsilon}\right| = 1$, but it can be defined as an improper integral,

$$I(\varepsilon) = \lim_{R\to\infty} \int_{-R}^{R} e^{it^2/\varepsilon}\, dt.$$

This convergence follows from an integration by parts:

$$\int_1^R e^{it^2/\varepsilon}\, dt = \Big[\frac{\varepsilon}{2it} e^{it^2/\varepsilon}\Big]_1^R + \int_1^R \frac{\varepsilon}{2it^2} e^{it^2/\varepsilon}\, dt.$$

The integrand oscillates rapidly away from the stationary phase point $t = 0$, and these parts contribute terms that are smaller than any power of ε, as we show below. The first oscillation near $t = 0$, where cancelation does not occur, has width of the order $\varepsilon^{1/2}$, so we expect that $I(\varepsilon) = O(\varepsilon^{1/2})$ as $\varepsilon \to 0$.

In fact, using contour integration and changing variables $t \mapsto e^{i\pi/4}s$ if $\varepsilon > 0$ and $t \mapsto E^{-i\pi/4}s$ if $\varepsilon < 0$, one can show that

$$\int_{-\infty}^{\infty} e^{it^2/\varepsilon}\, dt = \begin{cases} e^{i\pi/4}\sqrt{2\pi|\varepsilon|} & \text{if } \varepsilon > 0 \\ e^{-i\pi/4}\sqrt{2\pi|\varepsilon|} & \text{if } \varepsilon < 0 \end{cases}.$$

Next, we consider the integral

$$I(\varepsilon) = \int_{-\infty}^{\infty} f(t) e^{i\varphi(t)/\varepsilon}\, dt, \tag{3.9}$$

where $f : \mathbb{R} \to \mathbb{C}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ are smooth functions. A point $t = c$ is a stationary phase point if $\varphi'(c) = 0$. We call the stationary phase point nondegenerate if $\varphi''(c) \neq 0$.

Suppose that I has a single stationary phase point at $t = c$, which is nondegenerate. (If there are several such points, we simply add together the contributions from each one.) Then, using the idea that only the part of the integrand near the stationary phase point $t = c$ contributes significantly, we can Taylor expand the function f and the phase φ to approximate $I(\varepsilon)$ as follows:

$$I(\varepsilon) \quad \sim \quad \int f(c) \exp \frac{i}{\varepsilon}\left[\varphi(c) + \frac{1}{2}\varphi''(c)(t-c)^2\right] dt$$

$$\begin{aligned} &\sim f(c)e^{i\varphi(c)/\varepsilon}\int \exp\left[\frac{i\varphi''(c)}{2\varepsilon}s^2\right]\,ds \\ &\sim \sqrt{\frac{2\pi\varepsilon}{|\varphi''(c)|}}f(c)e^{i\varphi(c)/\varepsilon+i\sigma\pi/4}, \end{aligned}$$

where

$$\sigma = \operatorname{sgn}\varphi''(c).$$

More generally, we consider the asymptotic behavior as $\varepsilon \to 0$ of an integral of the form

$$I(x,\varepsilon) = \int A(x,\xi)e^{i\varphi(x,\xi)/\varepsilon}\,d\xi, \tag{3.10}$$

where $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$. We assume that

$$\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \qquad A : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}$$

are smooth (C^∞) functions, and that the support of A,

$$\operatorname{supp} A = \overline{\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m \mid A(x,\xi) \neq 0\}},$$

is a compact subset of $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 3.6 A stationary, or critical, point of the phase φ is a point $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\frac{\partial\varphi}{\partial\xi}(x,\xi) = 0. \tag{3.11}$$

A stationary phase point is nondegenerate if

$$\frac{\partial^2\varphi}{\partial\xi^2} = \left(\frac{\partial^2\varphi}{\partial\xi_i\partial\xi_j}\right)_{i,j=1,\ldots,m}$$

is invertible at the stationary phase point.

Proposition 3.7 If the support of A contains no stationary points of φ, then

$$I(x,\varepsilon) = O\left(\varepsilon^n\right) \qquad \text{as } \varepsilon \to 0$$

for every $n \in \mathbb{N}$.

Proof. Rewriting the integral in (3.10), and integrating by parts, we have

$$\begin{aligned} I(x,\varepsilon) &= -i\varepsilon\int A\frac{\partial\varphi}{\partial\xi}\cdot\frac{\partial}{\partial\xi}\left[e^{i\varphi/\varepsilon}\right]\left|\frac{\partial\varphi}{\partial\xi}\right|^{-2}\,d\xi \\ &= i\varepsilon\int \frac{\partial}{\partial\xi}\cdot\left[A\left|\frac{\partial\varphi}{\partial\xi}\right|^{-2}\frac{\partial\varphi}{\partial\xi}\right]e^{i\varphi/\varepsilon}\,d\xi \\ &= O(\varepsilon). \end{aligned}$$

Applying this argument n times, we get the result. $\square$

The implicit function theorem implies there is a unique local smooth solution of (3.11) for ξ in a neighborhood $U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$ We write this stationary phase point as as $\xi = \overline{\xi}(x)$, where $\overline{\xi} : U \to V$. We may reduce the case of multiple nondegenerate critical points to this one by means of a partition of unity, and may also suppose that $\operatorname{supp} A \subset U \times V$. According to the Morse lemma, there is a local change of coordinates $\xi \mapsto \eta$ near a nondegenerate critical point such that

$$\varphi(x,\xi) = \varphi\left(x, \overline{\xi}(x)\right) + \frac{1}{2}\frac{\partial^2 \varphi}{\partial \xi^2}\left(x, \overline{\xi}(x)\right) \cdot (\eta, \eta).$$

Making this change of variables in (3.10), and evaluating the resulting Fresnel integral, we get the following stationary phase formula [10].

Theorem 3.8 Let $I(x,\varepsilon)$ be defined by (3.10), where φ is a smooth real-valued function with a nondegenerate stationary phase point at $(x, \overline{\xi}(x))$, and A is a compactly supported smooth function whose support is contained in a sufficiently small neighborhood of the stationary phase point. Then, as $\varepsilon \to 0$,

$$I(x,\varepsilon) \sim \frac{(2\pi\varepsilon)^{n/2}}{\sqrt{\det \left|\frac{\partial^2 \varphi}{\partial \xi^2}\right|_{\xi=\overline{\xi}(x)}}} e^{i\varphi(x,\overline{\xi}(x))/\varepsilon + i\pi\sigma/4} \sum_{p=0}^{\infty} (i\varepsilon)^p R_p(x),$$

where

$$\sigma = \operatorname{sgn}\left(\frac{\partial^2 \varphi}{\partial \xi^2}\right)_{\xi=\overline{\xi}(x)}$$

is the signature of the matrix (the difference between the number of positive and negative eigenvalues), $R_0 = 1$, and

$$R_p(x) = \sum_{|k| \leq 2p} R_{pk}(x) \left.\frac{\partial^k A}{\partial \xi^k}\right|_{\xi=\overline{\xi}(x)},$$

where the R_{pk} are smooth functions depending only on φ.

3.4 Airy functions and degenerate stationary phase points

The behavior of the integral in (3.10) is more complicated when it has degenerate stationary phase points. Here, we consider the simplest case, where $\xi \in \mathbb{R}$ and two stationary phase points coalesce. The asymptotic behavior of the integral in a neighborhood of the degenerate critical point is then described by an Airy function.

Airy functions are solutions of the ODE

$$y'' = xy. \tag{3.12}$$

The behavior of these functions is oscillatory as $x \to -\infty$ and exponential as $x \to \infty$. They are the most basic functions that exhibit a transition from oscillatory to exponential behavior, and because of this they arise in many applications (for example, in describing waves at caustics or turning points).

Two linearly independent solutions of (3.12) are denoted by $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$. The solution $\mathrm{Ai}(x)$ is determined up to a constant normalizing factor by the condition that $\mathrm{Ai}(x) \to 0$ as $x \to \infty$. It is conventionally normalized so that

$$\mathrm{Ai}(0) = \frac{1}{3^{2/3}}\Gamma\left(\frac{2}{3}\right),$$

where Γ is the Gamma-function. This solution decays exponentially as $x \to \infty$ and oscillates, with algebraic decay, as $x \to -\infty$ [16],

$$\mathrm{Ai}(x) \sim \begin{cases} \frac{1}{2}\pi^{-1/2}x^{-1/4}\exp[-2x^{3/2}/3] & \text{as } x \to \infty, \\ \pi^{-1/2}(-x)^{-1/4}\sin[2(-x)^{3/2}/3 + \pi/4] & \text{as } x \to -\infty. \end{cases}$$

The solution $\mathrm{Bi}(x)$ grows exponentially as $x \to \infty$.

We can derive these results from an integral representation of $\mathrm{Ai}(x)$ that is obtained by taking the Fourier transform of (3.12).* Let $\widehat{y} = \mathcal{F}[y]$ denote the Fourier transform of y,

$$\begin{aligned} \widehat{y}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} y(x)e^{-ikx}\,dx, \\ y(x) &= \int_{-\infty}^{\infty} \widehat{y}(k)e^{ikx}\,dk. \end{aligned}$$

Then

$$\mathcal{F}[y''] = -k^2\widehat{y}, \quad \mathcal{F}[-ixy] = \widehat{y}'.$$

Fourier transforming (3.12), we find that

$$-k^2\widehat{y} = i\widehat{y}'.$$

Solving this first-order ODE, we get

$$\widehat{y}(k) = ce^{ik^3/3},$$

so y is given by the oscillatory integral

$$y(x) = c\int_{-\infty}^{\infty} e^{i\left(kx+k^3/3\right)}\,dk.$$

*We do not obtain Bi by this method because it grows exponentially as $x \to \infty$, which is too fast for its Fourier transform to be well-defined, even as a tempered distribution.

The standard normalization for the Airy function corresponds to $c = 1/(2\pi)$, and thus

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\left(kx+k^3/3\right)}\, dk. \tag{3.13}$$

This oscillatory integral is not absolutely convergent, but it can be interpreted as the inverse Fourier transform of a tempered distribution. The inverse transform is a C^∞ function that extends to an entire function of a complex variable, as can be seen by shifting the contour of integration upwards to obtain the absolutely convergent integral representation

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty+i\eta}^{\infty+i\eta} e^{i\left(kx+k^3/3\right)}\, dk.$$

Just as the Fresnel integral with a quadratic phase, provides an approximation near a nondegenerate stationary phase point, the Airy integral with a cubic phase provides an approximation near a degenerate stationary phase point in which the third derivative of the phase in nonzero. This occurs when two nondegenerate stationary phase points coalesce.

Let us consider the integral

$$I(x,\varepsilon) = \int_{-\infty}^{\infty} f(x,t) e^{i\varphi(x,t)/\varepsilon}\, dt.$$

Suppose that we have nondegenerate stationary phase points at

$$t = t_\pm(x)$$

for $x < x_0$, which are equal when $x = x_0$ so that $t_\pm(x_0) = t_0$. We assume that

$$\varphi_t\left(x_0,t_0\right) = 0, \quad \varphi_{tt}\left(x_0,t_0\right) = 0, \quad \varphi_{ttt}\left(x_0,t_0\right) \neq 0.$$

Then Chester, Friedman, and Ursell (1957) showed that in a neighborhood of (x_0,t_0) there is a local change of variables $t = \tau(x,s)$ and functions $\psi(x)$, $\rho(x)$ such that

$$\varphi(x,t) = \psi(x) + \rho(x)s + \frac{1}{3}s^3.$$

Here, we have $\tau(x_0,0) = t_0$ and $\rho(x_0) = 0$. The stationary phase points correspond to $s = \pm\sqrt{-\rho(x)}$, where $\rho(x) < 0$ for $x < x_0$.

Since the asymptotic behavior of the integral as $\varepsilon \to 0$ is dominated by the contribution from the neighborhood of the stationary phase point, we expect that

$$\begin{aligned} I(x,\varepsilon) &\sim \int_{-\infty}^{\infty} f\left(x,\tau(x,s)\right)\tau_s(x,s) e^{i\left[\psi(x)+\rho(x)s+\frac{1}{3}s^3\right]/\varepsilon}\, ds \\ &\sim f\left(x_0,t_0\right)\tau_s(x_0,0) e^{i\psi(x)/\varepsilon} \int_{-\infty}^{\infty} e^{i\left[\rho(x)s+\frac{1}{3}s^3\right]/\varepsilon}\, ds \end{aligned}$$

$$
\begin{aligned}
&\sim \quad \varepsilon^{1/3} f\left(x_0, t_0\right) \tau_s(x_0, 0) e^{i\psi(x)/\varepsilon} \int_{-\infty}^{\infty} e^{i\left[\varepsilon^{-2/3}\rho(x)k + \frac{1}{3}k^3\right]}\, dk \\
&\sim \quad 2\pi\varepsilon^{1/3} f\left(x_0, t_0\right) \tau_s(x_0, 0) e^{i\psi(x)/\varepsilon} \mathrm{Ai}\left(\frac{\rho(x)}{\varepsilon^{2/3}}\right),
\end{aligned}
$$

where we have made the change of variables $s = \varepsilon^{1/3}k$, and used the definition of the Airy function.

More generally, we have the following result. For the proof, see [10].

Theorem 3.9 Let $I(x, \varepsilon)$ be defined by (3.10), where $\varphi(x, \xi)$, with $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}$, is a smooth, real-valued function with a degenerate stationary phase point at $(x, \overline{\xi}(x))$. Suppose that

$$
\frac{\partial \varphi}{\partial \xi} = 0, \quad \frac{\partial^2 \varphi}{\partial \xi^2} = 0, \quad \frac{\partial^3 \varphi}{\partial \xi^3} \neq 0,
$$

at $\xi = \overline{\xi}(x)$, and $A(x, \xi)$ is a smooth function whose support is contained in a sufficiently small neighborhood of the degenerate stationary phase point. Then there are smooth real-valued functions $\psi(x)$, $\rho(x)$, and smooth functions $A_k(x)$, $B_k(x)$ such that

$$
I(x, \varepsilon) \sim \left[\varepsilon^{1/3} \mathrm{Ai}\left(\frac{\rho(x)}{\varepsilon^{2/3}}\right) \sum_{k=0}^{\infty} A_k(x) + i\varepsilon^{2/3} \mathrm{Ai}'\left(\frac{\rho(x)}{\varepsilon^{2/3}}\right) \sum_{k=0}^{\infty} B_k(x)\right] e^{i\psi(x)/\varepsilon}
$$

as $\varepsilon \to 0$.

3.4.1 *Dispersive wave propagation*

An important application of the method of stationary phase concerns the long-time, or large-distance, behavior of linear dispersive waves. Kelvin (1887) originally developed the method for this purpose, following earlier work by Cauchy, Stokes, and Riemann. He used it to study the pattern of dispersive water waves generated by a ship in steady motion, and showed that at large distances from the ship the waves form a wedge with a half-angle of $\sin^{-1}(1/3)$, or approximately $19.5°$.

As a basic example of a dispersive wave equation, we consider the following IVP (initial value problem) for the linearized KdV (Korteweg-de Vries), or Airy, equation,

$$
\begin{aligned}
&u_t = u_{xxx}, \\
&u(x, 0) = f(x).
\end{aligned}
$$

This equation provides an asymptotic description of linear, unidirectional, weakly dispersive long waves; for example, shallow water waves.

We assume for simplicity that the initial data $f : \mathbb{R} \to \mathbb{R}$ is a Schwarz function, meaning that it is smooth and decays, together with all its derivatives, faster than any polynomial as $|x| \to \infty$.

We use $\widehat{u}(k,t)$ to denote the Fourier transform of $u(x,t)$ with respect to x,

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{u}(k,t) e^{ikx}\, dk,$$
$$\widehat{u}(k,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx.$$

Then $\widehat{u}(k,t)$ satisfies

$$\widehat{u}_t + ik^3 \widehat{u} = 0,$$
$$\widehat{u}(k,0) = \widehat{f}(k).$$

The solution of this equation is

$$\widehat{u}(k,t) = \widehat{f}(k) e^{-i\omega(k)t},$$

where

$$\omega(k) = k^3.$$

The function $\omega : \mathbb{R} \to \mathbb{R}$ gives the (angular) frequency $\omega(k)$ of a wave with wavenumber k, and is called the *dispersion relation* of the KdV equation.

Inverting the Fourier transform, we find that the solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{f}(k) e^{ikx - i\omega(k)t}\, dk.$$

Using the convolution theorem, we can write this solution as

$$u(x,t) = f * g(x,t),$$

where the star denotes convolution with respect to x, and

$$g(x,t) = \frac{1}{(3t)^{1/3}} \mathrm{Ai}\left(-\frac{x}{(3t)^{1/3}} \right)$$

is the Green's function of the Airy equation.

We consider the asymptotic behavior of this solution as $t \to \infty$ with $x/t = v$ fixed. This limit corresponds to the large-time limit in a reference frame moving with velocity v. Thus, we want to find the behavior as $t \to \infty$ of

$$u(vt,t) = \int_{-\infty}^{\infty} \widehat{f}(k) e^{i\varphi(k,v)t}\, dk, \tag{3.14}$$

where

$$\varphi(k,v) = kv - \omega(k).$$

The stationary phase points satisfy $\varphi_k = 0$, or

$$v = \omega'(k).$$

The solutions are the wavenumbers k whose group velocity $\omega'(k)$ is equal to v. It follows that

$$3k^2 = v.$$

If $v < 0$, then there are no stationary phase points, and $u(vt, t) = o(t^{-n})$ as $t \to \infty$ for any $n \in \mathbb{N}$.

If $v > 0$, then there are two nondegenerate stationary phase points at $k = \pm k_0(v)$, where

$$k_0(v) = \sqrt{\frac{v}{3}}.$$

These two points contribute complex conjugate terms, and the method of stationary phase implies that

$$u(vt, t) \sim \sqrt{\frac{2\pi}{|\omega''(k_0)|t}}\widehat{f}(k_0)e^{i\varphi(k_0,v)t - i\pi/4} + \text{c.c.} \qquad \text{as } t \to \infty.$$

The energy in the wave-packet therefore propagates at the group velocity $C = \omega'(k)$,

$$C = 3k^2,$$

rather than the phase velocity $c = \omega/k$,

$$c = k^2.$$

The solution decays at a rate of $t^{-1/2}$, in accordance with the linear growth in t of the length of the wavetrain and the conservation of energy,

$$\int_{-\infty}^{\infty} u^2(x, t)\, dt = \text{constant}.$$

The two stationary phase points coalesce when $v = 0$, and then there is a single degenerate stationary phase point. To find the asymptotic behavior of the solution when v is small, we make the change of variables

$$k = \frac{\xi}{(3t)^{1/3}}$$

in the Fourier integral solution (3.14), which gives

$$u(x, t) = \frac{1}{(3t)^{1/3}} \int_{-\infty}^{\infty} \widehat{f}\left(\frac{\xi}{(3t)^{1/3}}\right) e^{-i\left(\xi w + \frac{1}{3}\xi^3\right)}\, d\xi,$$

where

$$w = -\frac{t^{2/3}v}{3^{1/3}}.$$

It follows that as $t \to \infty$ with $t^{2/3}v$ fixed,

$$u(x,t) \sim \frac{2\pi}{(3t)^{1/3}} \widehat{f}(0) \mathrm{Ai}\left(-\frac{t^{2/3}v}{3^{1/3}}\right).$$

Thus the transition between oscillatory and exponential behavior is described by an Airy function. Since $v = x/t$, the width of the transition layer is of the order $t^{1/3}$ in x, and the solution in this region is of the order $t^{-1/3}$. Thus it decays more slowly and is larger than the solution elsewhere.

Whitham [20] gives a detailed discussion of linear and nonlinear dispersive wave propagation.

3.5 Laplace's Method

Consider an integral

$$I(\varepsilon) = \int_{-\infty}^{\infty} f(t) e^{\varphi(t)/\varepsilon}\, dt,$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{C}$ are smooth functions, and ε is a small positive parameter. This integral differs from the stationary phase integral in (3.9) because the argument of the exponential is real, not imaginary. Suppose that φ has a global maximum at $t = c$, and the maximum is nondegenerate, meaning that $\varphi''(c) < 0$. The dominant contribution to the integral comes from the neighborhood of $t = c$, since the integrand is exponentially smaller in ε away from that point. Taylor expanding the functions in the integrand about $t = c$, we expect that

$$\begin{aligned} I(\varepsilon) &\sim \int f(c) e^{\left[\varphi(c) + \frac{1}{2}\varphi''(c)(t-c)^2\right]/\varepsilon}\, dt \\ &\sim f(c) e^{\varphi(c)/\varepsilon} \int_{-\infty}^{\infty} e^{\frac{1}{2}\varphi''(c)(t-c)^2/\varepsilon}\, dt. \end{aligned}$$

Using the standard integral

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}at^2}\, dt = \sqrt{\frac{2\pi}{a}},$$

we get

$$I(\varepsilon) \sim f(c) \left(\frac{2\pi\varepsilon}{|\varphi''(c)|}\right)^{1/2} e^{\varphi(c)/\varepsilon} \quad \text{as } \varepsilon \to 0^+.$$

This result can proved under suitable assumptions on f and φ, but we will not give a detailed proof here (see [17], for example).

Example 3.10 The Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

Integration by parts shows that if $n \in \mathbb{N}$, then

$$\Gamma(n+1) = n!.$$

Thus, the Gamma function extends the factorial function to arbitrary positive real numbers. In fact, the Gamma function can be continued to an analytic function

$$\Gamma : \mathbb{C} \setminus \{0, -1, -2, \ldots\} \to \mathbb{C}$$

with simple poles at $0, -1, -2, \ldots$.

Making the change of variables $t = xs$, we can write the integral representation of Γ as

$$\Gamma(x) = x^x \int_0^\infty \frac{1}{s} e^{x\varphi(s)}\, ds,$$

where

$$\varphi(s) = -s + \log s.$$

The phase $\varphi(s)$ has a nondegenerate maximum at $s = 1$, where $\varphi(1) = -1$, and $\varphi''(1) = -1$. Using Laplace's method, we find that

$$\Gamma(x) \sim \left(\frac{2\pi}{x}\right)^{1/2} x^x e^{-x} \quad \text{as } x \to \infty.$$

In particular, setting $x = n + 1$, and using the fact that

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e,$$

we obtain Stirling's formula for the factorial,

$$n! \sim (2\pi)^{1/2} n^{n+1/2} e^{-n} \quad \text{as } n \to \infty.$$

This expansion of the Γ-function can be continued to higher orders to give:

$$\Gamma(x) \sim \left(\frac{2\pi}{x}\right)^{1/2} x^x e^{-x} \left[1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \ldots\right] \quad \text{as } x \to \infty,$$
$$a_1 = \frac{1}{12}, \quad a_2 = \frac{1}{288}, \quad a_3 = -\frac{139}{51,840}, \quad \ldots.$$

3.5.1 *Multiple integrals*

Proposition 3.11 Let A be a positive-definite $n \times n$ matrix. Then

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}x^T Ax}\, dx = \frac{(2\pi)^{n/2}}{|\det A|^{1/2}}.$$

Proof. Since A is positive-definite (and hence symmetric) there is an orthogonal matrix S and a positive diagonal matrix $D = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$ such that

$$A = S^T DS.$$

We make the change of variables $y = Sx$. Since S is orthogonal, we have $|\det S| = 1$, so the Jacobian of this transformation is 1. We find that

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-\frac{1}{2}x^T Ax}\, dx &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}y^T Dy}\, dy \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}} e^{-\frac{1}{2}\lambda_i y_i^2}\, dy_i \\
&= \frac{(2\pi)^{n/2}}{(\lambda_1 \ldots \lambda_n)^{1/2}} \\
&= \frac{(2\pi)^{n/2}}{|\det A|^{1/2}}.
\end{aligned}$$

□

Now consider the multiple integral

$$I(\varepsilon) = \int_{\mathbb{R}^n} f(t) e^{\varphi(t)/\varepsilon}\, dt.$$

Suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}$ has a nondegenerate global maximum at $t = c$. Then

$$\varphi(t) = \varphi(c) + \frac{1}{2} D^2\varphi(c) \cdot (t - c, t - c) + O(|t - c|^3) \quad \text{as } t \to c.$$

Hence, we expect that

$$I(\varepsilon) \sim \int_{\mathbb{R}^n} f(c) e^{\left[\varphi(c) + \frac{1}{2}(t-c)^T A(t-c)\right]/\varepsilon}\, dt,$$

where A is the matrix of $D^2\varphi(c)$, with components

$$A_{ij} = \frac{\partial^2 \varphi}{\partial t_i \partial t_j}(c).$$

Using the previous proposition, we conclude that

$$I(\varepsilon) \sim \frac{(2\pi)^{n/2}}{|\det D^2\varphi(c)|^{1/2}} f(c) e^{\varphi(c)/\varepsilon}.$$

3.6 The method of steepest descents

Consider a contour integral of the form

$$I(\lambda) = \int_C f(z) e^{\lambda h(z)}\, dz,$$

where C is a contour in the complex plane, and $f, h : \mathbb{C} \to \mathbb{C}$ are analytic functions.

If $h(x+iy) = \varphi(x,y) + i\psi(x,y)$ is analytic, then $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ have no maxima or minima, and critical points where $h'(z) = 0$ are saddle points of φ, ψ. The curves $\varphi =$ constant, $\psi =$ constant are orthogonal except at critical points.

The idea of the method of steepest descents is to deform the contour C into a steepest descent contour passing through a saddle point on which φ has a maximum and $\psi =$ constant, so the contour is orthogonal to the level curves of φ. We then apply Laplace's method to the resulting integral. We will illustrate this idea by deriving the asymptotic behavior of the Airy function, given by (3.13)

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\left(kx + k^3/3\right)}\, dk.$$

To obtain the asymptotic behavior of $\mathrm{Ai}(x)$ as $x \to \infty$, we put this integral representation in a form that is suitable for the method of steepest descents. Setting $k = x^{1/2} z$, we find that

$$\mathrm{Ai}(x) = \frac{1}{2\pi} x^{1/2} I\left(x^{3/2}\right),$$

where

$$I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda\left[z + \frac{1}{3}z^3\right]}\, dz.$$

The phase

$$h(z) = i\left(z + \frac{1}{3}z^3\right)$$

has critical points at $z = \pm i$.

Writing $h = \varphi + i\psi$ in terms of its real and imaginary parts, we have

$$\varphi(x,y) = -y\left(1 + x^2 - \frac{1}{3}y^2\right),$$
$$\psi(x,y) = x\left(1 + \frac{1}{3}x^2 - y^2\right).$$

The steepest descent contour $\psi(x,y) = 0$ through $z = i$, or $(x,y) = (0,1)$, is

$$y = \sqrt{1 + \frac{1}{3}x^2}.$$

When $\lambda > 0$, we can deform the integration contour $(-\infty, \infty)$ upwards to this steepest descent contour C, since the integrand decays exponentially as $|z| \to \infty$ in the upper-half plane. Thus,

$$I(\lambda) = \int_C e^{i\lambda\left[z+\frac{1}{3}z^3\right]}\, dz.$$

We parameterize C by $z(t) = x(t) + iy(t)$, where

$$x(t) = \sqrt{3}\sinh t, \quad y(t) = \cosh t.$$

Then we find that

$$I(\lambda) = \int_{-\infty}^{\infty} f(t) e^{i\lambda\varphi(t)}\, dt,$$

where

$$\begin{aligned} f(t) &= \sqrt{3}\cosh t + i\sinh t, \\ \varphi(t) &= \cosh t\left[2 - \frac{8}{3}\cosh^2 t\right]. \end{aligned}$$

The maximum of $\varphi(t)$ occurs at $t = 0$, where

$$\varphi(0) = -2/3, \quad \varphi'(0) = 0, \quad \varphi''(0) = -6.$$

Laplace's method implies that

$$\begin{aligned} I(\lambda) &\sim f(0)\left(\frac{2\pi}{-\lambda\varphi''(0)}\right)^{1/2} e^{\lambda\varphi(0)} \\ &\sim \left(\frac{\pi}{\lambda}\right)^{1/2} e^{-2\lambda/3}. \end{aligned}$$

It follows that

$$\mathrm{Ai}(x) \sim \frac{1}{2\pi^{1/2}x^{1/4}} e^{-2x^{3/2}/3} \qquad \text{as } x \to \infty. \tag{3.15}$$

Using the method of stationary phase, one can show from (3.13) that the asymptotic behavior of the Airy function as $x \to -\infty$ is given by

$$\mathrm{Ai}(x) \sim \frac{1}{\pi^{1/2}|x|^{1/4}} \sin\left[\frac{2}{3}|x|^{3/2} + \frac{\pi}{4}\right]. \tag{3.16}$$

This result is an example of a connection formula. It gives the asymptotic behavior as $x \to -\infty$ of the solution of the ODE (3.12) that decays exponentially as $x \to \infty$. This connection formula is derived using the integral representation (3.13), which provides global information about the solution.

Chapter 4

The Method of Matched Asymptotic Expansions: ODEs

Many singularly perturbed differential equations have solutions that change rapidly in a narrow region. This may occur in an initial layer where there is a rapid adjustment of initial conditions to a quasi-steady state, in a boundary layer where the solution away from the boundary adjusts to a boundary condition, or in an interior layer such as a propagating wave-front.

These problems can be analyzed using the method of matched asymptotic expansions (MMAE), in which we construct different asymptotic solutions inside and outside the region of rapid change, and 'match' them together to determine a global solution. A typical feature of this type of problem is a reduction in the order of the differential equation in the unperturbed problem, leading to a reduction in the number of initial or boundary conditions that can be imposed upon the solution. For additional information, see [3], [19].

4.1 Enzyme kinetics

Enzymes are proteins that act as catalysts. (There are also a smaller number of enzymes, called ribozymes, that contain catalytic RNA molecules.) A substance that is acted upon by an enzyme to create a product is called a substrate. Enzymes are typically very specific and highly efficient catalysts — tiny concentrations of enzymes compared with substrate concentrations are required.

For example, the enzyme catalase catalyzes the decomposition of hydrogen peroxide into water and oxygen, and one molecule of catalase can break up 40 million molecules of hydrogen peroxide each second. As another example, carbonic anhydrase occurs in red blood cells, where it catalyzes the reaction $CO_2 + H_2O \leftrightarrow H_2CO_3$ that enables the cells to transport carbon dioxide from the tissues to the lungs. One molecule of carbonic anhydrase can process one million molecules of CO_2 each second.

Michaelis and Menton (1913) proposed a simple model of enzymatically controlled reactions, in which the enzyme E and substrate S combine to form a com-

plex C, and the complex breaks down irreversibly into the enzyme and a product P. Symbolically, we have

$$E + S \longleftrightarrow C \longrightarrow E + P.$$

We let

$$e = [E], \quad s = [S], \quad c = [C], \quad p = [P],$$

denote the concentrations of the corresponding species.

According to the law of mass action, the rate of a reaction is proportional to the product of the concentrations of the species involved, so that

$$\begin{aligned}
\frac{de}{dt} &= -k_0 es + (k_0 + k_2)\, c, \\
\frac{ds}{dt} &= -k_1 es + k_0 c, \\
\frac{dc}{dt} &= k_1 es - (k_0 + k_2)\, c, \\
\frac{dp}{dt} &= k_2 c,
\end{aligned}$$

where k_0, k_1, k_2 are rate constants. We impose initial conditions

$$s(0) = s_0, \quad e(0) = e_0, \quad c(0) = c_0, \quad p(0) = 0,$$

corresponding to an initial state with substrate and enzyme but no complex or product.

The equation for p decouples from the remaining equations. Adding the equations for e and c, we get

$$\frac{d}{dt}\,(e + c) = 0,$$

which implies that

$$e(t) + c(t) = e_0.$$

Thus, the equations reduce to a pair of ODEs for s and c:

$$\begin{aligned}
\frac{ds}{dt} &= -k_1 e_0 s + (k_1 s + k_0)\, c, \\
\frac{dc}{dt} &= k_1 e_0 s - (k_1 s + k_0 + k_2)\, c, \\
s(0) &= s_0, \qquad c(0) = 0.
\end{aligned}$$

We introduce dimensionless quantities

$$u(\tau) = \frac{s(t)}{s_0}, \quad v(\tau) = \frac{c(t)}{e_0}, \quad \tau = k_1 e_0 t,$$

$$\lambda = \frac{k_2}{k_1 s_0}, \quad k = \frac{k_0 + k_2}{k_1 s_0}, \quad \varepsilon = \frac{e_0}{s_0}.$$

Then u, v satisfy

$$\begin{aligned} \frac{du}{d\tau} &= -u + (u + k - \lambda)v, \\ \varepsilon \frac{dv}{d\tau} &= u - (u + k)v, \\ u(0) &= 1, \qquad v(0) = 0, \end{aligned} \tag{4.1}$$

where $\lambda > 0$ and $k > \lambda$.

The enzyme concentration is typically much less than the substrate concentration, and the ratio ε is usually in the range 10^{-2} to 10^{-7}. Thus, we want to solve (4.1) when ε is small.

This is a singular perturbation problem because the order of the system drops by one when $\varepsilon = 0$, and we cannot impose an initial condition on the dimensionless complex concentration v. As we will see below, what happens is this: there is an initial rapid adjustment of the complex and enzyme concentrations to quasi-equilibrium values on a time-scale of the order ε. Then there is a slower conversion of the substrate into the product on a time-scale of the order 1. We will construct inner and outer solutions that describe these processes and match them together.

4.1.1 *Outer solution*

We look for a straightforward expansion of the form

$$\begin{aligned} u(\tau, \varepsilon) &= u_0(\tau) + \varepsilon u_1(\tau) + O(\varepsilon^2), \\ v(\tau, \varepsilon) &= v_0(\tau) + \varepsilon v_1(\tau) + O(\varepsilon^2). \end{aligned}$$

Using these expansion in (4.1), and equating the leading order terms of the order ε^0, we find that

$$\begin{aligned} \frac{du_0}{d\tau} &= -u_0 + (u_0 + k - \lambda)\, v_0, \\ 0 &= u_0 - (u_0 + k)\, v_0. \end{aligned}$$

We cannot impose both initial conditions on the leading-order outer solution. We will therefore take the most general solution of these equations. We will see, however, when we come to matching that the natural choice of imposing the initial condition $u_0(0) = 1$ is in fact correct.

From the second equation,

$$v_0 = \frac{u_0}{u_0 + k}.$$

This complex concentration v_0 corresponds to a quasi-equilibrium for the substrate concentration u_0, in which the creation of the complex by the binding of the enzyme

with the substrate is balanced by the destruction of the complex by the reverse reaction and the decomposition of the complex into the product and the enzyme. Substituting this result into the first equation, we get a first order ODE for $u_0(\tau)$:

$$\frac{du_0}{d\tau} = -\frac{\lambda u_0}{u_0 + k}.$$

The solution of this equation is given by

$$u_0(\tau) + k \log u_0(\tau) = a - \lambda\tau,$$

where a is a constant of integration. This solution is invalid near $\tau = 0$ because no choice of a can satisfy the initial conditions for both u_0 and v_0.

4.1.2 *Inner solution*

There is a short initial layer, for times $t = O(\varepsilon)$, in which u, v adjust from their initial values to values that are compatible with the outer solution found above. We introduce inner variables

$$T = \frac{\tau}{\varepsilon}, \quad U(T, \varepsilon) = u(\tau, \varepsilon), \quad V(T, \varepsilon) = v(\tau, \varepsilon).$$

The inner equations are

$$\begin{aligned}
&\frac{dU}{dT} = \varepsilon \left\{ -U + (U + k - \lambda)V \right\}, \\
&\frac{dV}{dT} = U - (U + k)V, \\
&U(0, \varepsilon) = 1, \qquad V(0, \varepsilon) = 0.
\end{aligned}$$

We look for an innner expansion

$$\begin{aligned}
U(T, \varepsilon) &= U_0(T) + \varepsilon U_1(T) + O(\varepsilon^2), \\
V(T, \varepsilon) &= V_0(T) + \varepsilon V_1(T) + O(\varepsilon^2).
\end{aligned}$$

The leading order inner equations are

$$\begin{aligned}
&\frac{dU_0}{dT} = 0, \\
&\frac{dV_0}{dT} = U_0 - (U_0 + k)V_0, \\
&U_0(0) = 1, \qquad V_0(0) = 0.
\end{aligned}$$

The solution is

$$\begin{aligned}
&U_0 = 1, \\
&V_0 = \frac{1}{1 + k} \left[1 - e^{-(1+k)T} \right].
\end{aligned}$$

4.1.3 *Matching*

We assume that the inner and outer expansions are both valid for intermediate times of the order $\varepsilon \ll \tau \ll 1$. We require that the expansions agree asymptotically in this regime, where $T \to \infty$ and $\tau \to 0$ as $\varepsilon \to 0$. Hence, the matching condition is

$$\lim_{T\to\infty} U_0(T) = \lim_{\tau\to 0^+} u_0(\tau),$$
$$\lim_{T\to\infty} V_0(T) = \lim_{\tau\to 0^+} v_0(\tau).$$

This condition implies that

$$u_0(0) = 1, \qquad v_0(0) = \frac{1}{1+k},$$

which is satisfied when $a = 1$ in the outer solution. Hence

$$u_0(\tau) + k \log u_0(\tau) = 1 - \lambda\tau.$$

The *slow manifold* for the enzyme system is the curve

$$v = \frac{u}{u+k}.$$

Trajectories rapidly approach the slow manifold in the initial layer. They then move more slowly along the slow manifold and approach the equilibrium $u = v = 0$ as $\tau \to \infty$. The inner layer corresponds to the small amount of enzyme 'loading up" on the substrate. The slow manifold corresponds to the enzyme working at full capacity in converting substrate into product.

A principle quantity of biological interest is the rate of uptake,

$$r_0 = \left.\frac{du_0}{d\tau}\right|_{\tau=0}.$$

It follows from the outer solution that

$$r_0 = \frac{\lambda}{1+k}.$$

The dimensional form of the rate of uptake is

$$\begin{aligned} R_0 &= \frac{ds}{dt} \\ &= \frac{Qs_0}{s_0 + k_m} \end{aligned}$$

where $Q = k_2 e_0$ is the maximum reaction rate, and

$$k_m = \frac{k_0 + k_2}{k_1}$$

is the Michaelis constant. The maximum rate depends only on k_2; the rate limiting step is $C \to P + E$.

For more information about enzyme kinetics, see [12].

4.2 General initial layer problems

Consider a dynamical system for $x(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^n$:

$$\begin{aligned} \dot{x} &= f(x, y), \\ \varepsilon \dot{y} &= g(x, y), \\ x(0) &= x_0, \qquad y(0) = y_0. \end{aligned} \tag{4.2}$$

Here, $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$. Suppose that there is a function $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$g\left(x, \varphi(x)\right) = 0,$$

and for each fixed $x \in \mathbb{R}^n$, the solution $y = \varphi(x)$ is a globally asymptotically stable equilibrium of the 'fast' system

$$\varepsilon \dot{y}(t) = g\left(x, y(t)\right). \tag{4.3}$$

Then the behavior of solutions of (4.2) is as follows:

(a) for $t = O(\varepsilon)$, there is a short initial layer in which $x(t)$ is nearly constant, and close to its initial value x_0, and $y(t)$ changes rapidly from its initial value to the quasi-steady state $y = \varphi(x_0)$.

(b) for $t = O(1)$, the solution is close to the slow manifold $y = \varphi(x) + O(\varepsilon)$, and $x(t)$ satisfies

$$\dot{x} = f\left(x, \varphi(x)\right).$$

If (4.3) does not have a unique globally stable equilibrium for every $x \in \mathbb{R}^m$, then more complex phenomena can occur.

An interesting example of a fast-slow system of ODEs arises in modeling the phenomenon of bursting in pancreatic β-cells. These cells are responsible for producing insulin which regulates glucose levels in the body. The β-cells are observed to undergo 'bursting' in which their membrane potential oscillates rapidly, with periods of the order of milliseconds. These oscillations stimulate the secretion of insulin by the cell. The length of each bursting period is on the order of seconds, and its length is influenced by the amount of glucose in the bloodstream. Thus, this mechanism provides one way that the body regulates glucose.

The basic mathematical model of bursting [12] consists of a fast/slow system. The fast system undergoes a Hopf bifurcation, corresponding to the appearance of a limit cycle oscillation, as the slow variable increases. On further increase in the slow variable the limit cycle disappears at a homoclinic bifurcation, and the fast system switches to a stable quasi-steady states. A decrease in the slow variable

leads to a saddle-node bifurcation which destroys this quasi-steady state. When the fast system is in its limit-cycle state, it drives an increase in the slow variable, and when the fast system is in its quasi-steady state it drives a decrease in the slow variable. The overall effect of this dynamics is a periodic oscillation of the slow variable on a long time scale which switches on and off the rapid periodic bursting of the fast system.

4.3 Boundary layer problems

The following explicitly solvable model boundary-value problem for a second order linear ODE illustrates the phenomenon of boundary layers:

$$\begin{aligned} &\varepsilon y'' + 2y' + y = 0, \quad 0 < x < 1, \\ &y(0) = 0, \qquad y(1) = 1. \end{aligned} \tag{4.4}$$

Here, the prime denotes a derivative with respect to x, and ε is a small positive parameter. The order of the ODE reduces from two to one when $\varepsilon = 0$, so we cannot expect to impose both boundary conditions on the solution. As we will see, when ε is small, there is a thin boundary layer (of width the order of ε) near $x = 0$ where the solution changes rapidly to take on the boundary value.

4.3.1 *Exact solution*

The exponential solutions of this equation are $y = e^{mx}$ where

$$m = \frac{-1 \pm \sqrt{1-\varepsilon}}{\varepsilon}.$$

We write these roots as $m = -\alpha, -\beta/\varepsilon$ where

$$\begin{aligned} \alpha(\varepsilon) &= \frac{1-\sqrt{1-\varepsilon}}{\varepsilon} \\ &= \frac{1}{2} + O(\varepsilon), \\ \beta(\varepsilon) &= 1 + \sqrt{1-\varepsilon} \\ &= 2 + O(\varepsilon). \end{aligned}$$

The general solution is

$$y(x,\varepsilon) = ae^{-\alpha(\varepsilon)x} + be^{-\beta(\varepsilon)x/\varepsilon}.$$

Imposing the boundary conditions and solving for the constants of integration a, b, we find that

$$y(x,\varepsilon) = \frac{e^{-\alpha x} - e^{-\beta x/\varepsilon}}{e^{-\alpha} - e^{-\beta/\varepsilon}}.$$

Thus, the solution involves two terms which vary on widely different length-scales.

Let us consider the behavior of this solution as $\varepsilon \to 0^+$. The asymptotic behavior is nonuniform, and there are two cases, which lead to matching outer and inner solutions.

(a) *Outer limit*: $x > 0$ fixed and $\varepsilon \to 0^+$. Then

$$y(x, \varepsilon) \to y_0(x),$$

where

$$y_0(x) = \frac{e^{-x/2}}{e^{-1/2}}. \tag{4.5}$$

This leading-order outer solution satisfies the boundary condition at $x = 1$ but not the boundary condition at $x = 0$. Instead, $y_0(0) = e^{1/2}$.

(b) *Inner limit*: $x/\varepsilon = X$ fixed and $\varepsilon \to 0^+$. Then

$$y(\varepsilon X, \varepsilon) \to Y_0(X),$$

where

$$Y_0(X) = \frac{1 - e^{-2X}}{e^{-1/2}}.$$

This leading-order inner solution satisfies the boundary condition at $x = 0$, but not the one at $x = 1$, which corresponds to $X = 1/\varepsilon$. Instead, we have $\lim_{X\to\infty} Y_0(X) = e^{1/2}$.

(c) *Matching*: Both the inner and outer expansions are valid in the region $\varepsilon \ll x \ll 1$, corresponding to $x \to 0$ and $X \to \infty$ as $\varepsilon \to 0$. They satisfy the matching condition

$$\lim_{x\to 0^+} y_0(x) = \lim_{X\to\infty} Y_0(X).$$

Let us construct an asymptotic solution of (4.4) without relying on the fact that we can solve it exactly.

4.3.2 *Outer expansion*

We begin with the outer solution. We look for a straightforward expansion

$$y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2).$$

We use this expansion in (4.4) and equate the coefficients of the leading-order terms to zero. Guided by our analysis of the exact solution, we only impose the boundary condition at $x = 1$. We will see later that matching is impossible if, instead, we

attempt to impose the boundary condition at $x = 0$. We obtain that

$$2y_0' + y_0 = 0,$$
$$y_0(1) = 1.$$

The solution is given by (4.5), in agreement with the expansion of the exact solution.

4.3.3 *Inner expansion*

Next, we consider the inner solution. We suppose that there is a boundary layer at $x = 0$ of width $\delta(\varepsilon)$, and introduce a stretched inner variable $X = x/\delta$. We look for an inner solution

$$Y(X, \varepsilon) = y(x, \varepsilon).$$

Since

$$\frac{d}{dx} = \frac{1}{\delta}\frac{d}{dX},$$

we find from (4.4) that Y satisfies

$$\frac{\varepsilon}{\delta^2}Y'' + \frac{2}{\delta}Y' + Y = 0,$$

where the prime denotes a derivative with respect to X. There are two possible dominant balances in this equation: (a) $\delta = 1$, leading to the outer solution; (b) $\delta = \varepsilon$, leading to the inner solution. Thus, we conclude that the boundary layer thickness is of the order ε, and the appropriate inner variable is

$$X = \frac{x}{\varepsilon}.$$

The equation for Y is then

$$Y'' + 2Y' + \varepsilon Y = 0,$$
$$Y(0, \varepsilon) = 0.$$

We impose only the boundary condition at $X = 0$, since we do not expect the inner expansion to be valid outside the boundary layer where $x = O(\varepsilon)$.

We seek an inner expansion

$$Y(X, \varepsilon) = Y_0(X) + \varepsilon Y_1(X) + O(\varepsilon^2)$$

and find that

$$Y_0'' + 2Y_0' = 0,$$
$$Y_0(0) = 0.$$

The general solution of this problem is

$$Y_0(X) = c\left[1 - e^{-2X}\right], \tag{4.6}$$

where c is an arbitrary constant of integration.

4.3.4 *Matching*

We can determine the unknown constant c in (4.6) by requiring that the inner solution matches with the outer solution (4.5). Here the matching condition is simply that

$$\lim_{x\to 0^+} y_0(x) = \lim_{X\to\infty} Y_0(X),$$

which implies that

$$c = e^{1/2}.$$

In summary, the asymptotic solution as $\varepsilon \to 0^+$, is given by

$$y(x,\varepsilon) = \begin{cases} e^{1/2}\left[1 - e^{-2x/\varepsilon}\right] & \text{as } \varepsilon \to 0^+ \text{ with } x/\varepsilon \text{ fixed,} \\ e^{-x/2+1/2} & \text{as } \varepsilon \to 0^+ \text{ with } x > 0 \text{ fixed.} \end{cases}$$

A more systematic way to match solutions, which is useful in problems where the behavior of the solution is not as simple, is to introduce an intermediate variable $\xi = x/\eta(\varepsilon)$, where $\varepsilon \ll \eta(\varepsilon) \ll 1$ as $\varepsilon \to 0^+$, and require that the inner and outer solutions have the same asymptotic behavior as $\varepsilon \to 0^+$ with ξ fixed for suitably chosen η. This requirement holds provided that both the inner and outer expansions hold in an intermediate 'overlap' region in which $x = O(\eta)$.

4.3.5 *Uniform solution*

We have constructed two different inner and outer asymptotic solutions in two different regions. Sometimes it is convenient to have a single uniform solution. This can be constructed from the inner and outer solutions as follows:

$$y_{\text{uniform}} = y_{\text{inner}} + y_{\text{outer}} - y_{\text{overlap}}.$$

Here, the function y_{overlap} is the common asymptotic behavior of the inner and outer solutions in the matching region. Inside the boundary layer, we have $y_{\text{outer}} \sim y_{\text{overlap}}$, so $y_{\text{uniform}} \sim y_{\text{inner}}$. Away from the boundary layer, we have $y_{\text{inner}} \sim y_{\text{overlap}}$, so $y_{\text{uniform}} \sim y_{\text{outer}}$. Thus, in either case the uniform solution y_{uniform} has the correct asymptotic behavior.

For the model ODE problem solved above, we have $y_{\text{overlap}} = e^{1/2}$, and the leading order uniform solution is given by

$$y_{\text{uniform}}(x,\varepsilon) = e^{1/2}\left[e^{-x/2} - e^{-2x/\varepsilon}\right].$$

There are systematic matching methods that provide higher-order matched asymptotic solutions, but we will not discuss them here. In general such expansions may not converge, reflecting the singular nature of the perturbation problem. This can

also be anticipated from the fact that the location of the boundary layer switches abruptly from $x = 0$ to $x = 1$ as the sign of ε switches from positive to negative.

4.3.6 *Why is the boundary layer at $x = 0$?*

Suppose we impose the boundary condition at $x = 0$ on the outer solution and look for an inner solution and a boundary layer at $x = 1$. The leading-order outer solution y_0 satisfies

$$\begin{aligned} 2y_0' + y_0 &= 0, \\ y_0(0) &= 0, \end{aligned}$$

so that

$$y_0(x) = 0.$$

We look for an inner expansion $y(x, \varepsilon) = Y(X, \varepsilon)$ in a boundary layer near $x = 1$, depending on a stretched inner variable

$$X = \frac{1 - x}{\varepsilon}.$$

The leading-order inner solution $Y_0(X) = Y(X, 0)$ satisfies

$$\begin{aligned} Y_0'' - 2Y_0' &= 0, \\ Y_0(0) &= 1. \end{aligned}$$

The solution is

$$Y_0(X) = e^{2X} + c.$$

In this case, the inner solution grows exponentially into to interior of the domain, and $Y_0(X) \to \infty$ as $X \to \infty$. Thus, no matching with the outer solution is possible.

4.4 Boundary layer problems for linear ODEs

Consider the linear BVP

$$\begin{aligned} &\varepsilon y'' + a(x)y' + b(x)y = 0 \qquad 0 < x < 1, \\ &y(0) = \alpha, \qquad y(1) = \beta, \end{aligned}$$

where $a, b : [0, 1] \to \mathbb{R}$ are continuous functions, and α, β are constants.

The requirement that the inner, boundary layer solution decays exponentially into the interior of the interval implies that if $a(0) > 0$, then a boundary layer can occur at $x = 0$, and if $a(1) < 0$, then a boundary layer can occur at $x = 1$. Thus, if a does not change sign on $[0, 1]$, the boundary layer can occur at only one end, while if a changes sign, then more complicated behavior is possible:

(a) if $a(x) > 0$, then the boundary layer is at $x = 0$;
(b) if $a(x) < 0$, then the boundary layer is at $x = 1$;
(c) if $a(x)$ changes sign and $a'(x) > 0$, then a boundary layer cannot occur at either endpoint (in this case a corner layer typically occurs in the interior);
(d) if $a(x)$ changes sign and $a'(x) < 0$, then a boundary layer can occur at both endpoints.

The first two cases are treated by a straightforward modification of the expansion for constant coefficients. The other two cases are more difficult, and we illustrate them with some examples.

Example 4.1 Consider the BVP

$$\begin{aligned} &\varepsilon y'' + xy' - y = 0 \quad -1 < x < 1, \\ &y(-1) = 1, \qquad y(1) = 2. \end{aligned}$$

This ODE can be solved exactly. One solution is $y(x) = x$. A second linearly independent solution can be found by reduction of order, which gives

$$y(x) = e^{-x^2/(2\varepsilon)} + \frac{x}{\varepsilon} \int e^{-x^2/(2\varepsilon)}\, dx.$$

We will use the MMAE to construct an asymptotic solution without relying on an exact solution.

The inner solution grows exponentially into the interior at either end, so we cannot construct a boundary layer solution. We use instead left and right outer solutions

$$y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2),$$

where

$$xy_0' - y_0 = 0.$$

As we will see, matching implies that the left and right outer solutions are valid in the intervals $(-1, 0)$ and $(0, 1)$, respectively. Imposing the boundary conditions at the left and right, we therefore get

$$y(x, \varepsilon) \sim \begin{cases} -x & \text{as } \varepsilon \to 0^+ \text{ with } -1 \le x < 0 \text{ fixed}, \\ 2x & \text{as } \varepsilon \to 0^+ \text{ with } 0 < x < 1 \text{ fixed}. \end{cases}$$

These outer solutions meet at $x = 0$, where the coefficient of y' in the ODE vanishes. The outer solution has a 'corner' at that point.

We seek an inner solution inside a corner layer about $x = 0$. To find the appropriate scalings, we introduce the inner variables

$$X = \frac{x}{\delta}, \qquad Y = \eta Y,$$

and use a dominant balance argument. The rescaled ODE is

$$\frac{\varepsilon}{\delta^2}Y'' + XY' - Y = 0.$$

The dominant balance for the inner solution occurs when $\delta = \varepsilon^{1/2}$, and all three terms are of the same order of magnitude. Matching the inner solution with the left and right outer solutions, we find that

$$\eta Y(X,\varepsilon) \sim \begin{cases} -\delta X & \text{as } X \to -\infty, \\ 2\delta X & \text{as } X \to \infty. \end{cases}$$

We therefore choose $\eta = \delta$.

The leading order inner solution is then given by

$$y(x,\varepsilon) \sim \varepsilon^{1/2} Y_0\left(\frac{x}{\varepsilon^{1/2}}\right),$$

where $Y_0(X)$ satisfies

$$\begin{aligned} &Y_0'' + XY_0' - Y_0 = 0, \\ &Y_0(X) \sim \begin{cases} -X & \text{as } X \to -\infty, \\ 2X & \text{as } X \to \infty. \end{cases} \end{aligned}$$

In this case, the ODE does not simplify at all; however, we obtain a canonical boundary value problem on $\mathbb{R}$ for matching the two outer solutions.

The solution of this inner problem is

$$Y_0(X) = -X + \frac{3}{\sqrt{2\pi}}\left[e^{-X^2/2} + X\int_{-\infty}^{X} e^{-t^2/2}\,dt\right],$$

and this completes the construction of the leading order asymptotic solution. (Other problems may lead to ODEs that require the use of special functions.)

Example 4.2 Consider the BVP

$$\begin{aligned} &\varepsilon y'' - xy' + y = 0 \quad -1 < x < 1, \\ &y(-1) = 1, \qquad y(1) = 2. \end{aligned}$$

The coefficients of y and y' have the opposite sign to the previous example, and we can find an inner, boundary layer solution at both $x = 0$ and $x = 1$.

The leading order outer solution $y(x,\varepsilon) \sim y_0(x)$ satisfies

$$-xy_0' + y_0 = 0,$$

with solution

$$y_0(x) = Cx,$$

where C is a constant of integration.

The inner solution near $x = -1$ is given by

$$y(x, \varepsilon) = Y\left(\frac{1+x}{\varepsilon}, \varepsilon\right),$$

where $Y(X, \varepsilon)$ satisfies

$$\begin{aligned} &Y'' + (1 - \varepsilon X)Y' + \varepsilon Y = 0, \\ &Y(0, \varepsilon) = 1. \end{aligned}$$

Expanding

$$Y(X, \varepsilon) = Y_0(X) + \varepsilon Y_1(X) + \ldots,$$

we find that the leading order inner solution $Y_0(X)$ satisfies

$$\begin{aligned} &Y_0'' + Y_0' = 0, \\ &Y_0(0) = 1. \end{aligned}$$

The solution is

$$Y_0(X) = 1 + A\left(1 - e^{-X}\right),$$

where A is a constant of integration.

The inner solution near $x = 1$ is given by

$$y(x, \varepsilon) = Z\left(\frac{1-x}{\varepsilon}, \varepsilon\right),$$

where $Z(X, \varepsilon)$ satisfies

$$\begin{aligned} &Z'' + (1 - \varepsilon X)Z' + \varepsilon Z = 0, \\ &Z(0, \varepsilon) = 2. \end{aligned}$$

Expanding

$$Z(X, \varepsilon) = Z_0(X) + \varepsilon Z_1(X) + \ldots,$$

we find that the leading order inner solution $Z_0(X)$ satisfies

$$\begin{aligned} &Z_0'' + Z_0' = 0, \\ &Z_0(0) = 2. \end{aligned}$$

The solution is

$$Z_0(X) = 2 + B\left(1 - e^{-X}\right),$$

where B is a constant of integration.

The leading order matching condition implies that

$$\lim_{X\to\infty} Y_0(X) = \lim_{x\to -1} y_0(x),$$
$$\lim_{X\to\infty} Z_0(X) = \lim_{x\to 1} y_0(x),$$

or

$$1 + A = -C, \qquad 2 + B = C.$$

We conclude that

$$A = -(1 + C), \qquad B = C - 2.$$

The constant C is not determined by the matching conditions. Higher order matching conditions also do not determine C. Its value ($C = 1/2$) depends on the interaction between the solutions in the boundary layers at either end, which involves exponentially small effects [13].

4.5 A boundary layer problem for capillary tubes

In view of the subtle boundary layer behavior that can occur for linear ODEs, it is not surprising that the solutions of nonlinear ODEs can behave in even more complex ways. Various nonlinear boundary layer problems for ODEs are discussed in [3], [13], [19]. Here we will discuss a physical example of a boundary layer problem: the rise of a liquid in a wide capillary tube. This problem was first analyzed by Laplace [15]; see also [5], [14].

4.5.1 *Formulation*

Consider an open capillary tube of cross-section $\Omega \subset \mathbb{R}^2$ that is placed vertically in an infinite reservoir of fluid (such as water). Surface tension causes the fluid to rise up the tube, and we would like to compute the equilibrium shape of the meniscus and how high the fluid rises.

According the the Laplace-Young theory, there is a pressure jump $[p]$ across a fluid interface that is proportional to the mean curvature κ of the interface:

$$[p] = \sigma\kappa.$$

The constant of proportionality σ is the coefficient of surface tension.

We use (x, y) as horizontal coordinates and z as a vertical coordinate, where we measure the height z from the undisturbed level of the liquid far away from the tube and pressure p from the corresponding atmospheric pressure. Then, assuming that the fluid is in hydrostatic equilibrium, the pressure of a column of fluid of height z is $\rho g z$, where ρ is the density of the fluid (assumed constant), and g is the acceleration due to gravity.

If the fluid interface is a graph $z = u(x, y)$, then its mean curvature is given by

$$\kappa = -\nabla \cdot \left[\frac{\nabla u}{\left(1 + |\nabla u|^2\right)^{1/2}} \right],$$

where ∇ denotes the derivative with respect to the horizontal coordinates. Choosing the sign of the pressure jump appropriately, we find that u satisfies the following PDE in Ω

$$\sigma \nabla \cdot \left[\frac{\nabla u}{\left(1 + |\nabla u|^2\right)^{1/2}} \right] = \rho g u.$$

The boundary condition for the PDE follows from the fact that the fluid makes a fixed angle θ_w, called the wetting angle, with the wall of the tube. Hence on the boundary $\partial\Omega$, we have

$$\frac{\partial u}{\partial n} = \tan \theta_0,$$

where $\theta_0 = \pi/2 - \theta_w$. For definiteness, we assume that $0 < \theta_0 < \pi/2$.

Let a be a typical length-scale of the tube cross-section (for example, the radius of a circular tube). We introduce dimensionless variables

$$u^* = \frac{u}{a}, \quad x^* = \frac{x}{a}, \quad y^* = \frac{y}{a}.$$

Then, after dropping the stars, we find that the nondimensionalized problem is

$$\begin{aligned} \varepsilon^2 \nabla \cdot \left[\frac{\nabla u}{\left(1 + |\nabla u|^2\right)^{1/2}} \right] &= u \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= \tan \theta_0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where

$$\varepsilon^2 = \frac{\sigma}{\rho g a^2}.$$

We define the capillary length-scale

$$\ell = \sqrt{\frac{\sigma}{\rho g}}.$$

This a characteristic length-scale for the balance between surface-tension and gravity forces, and we expect the fluid to rise up the tube by an amount of this order. We can write $\varepsilon = \ell/a$, meaning that it is the ratio of the capillary length-scale to the width of the tube. When $\varepsilon \gg 1$, we have a 'narrow' tube, and when $\varepsilon \ll 1$ we have a 'wide' tube.

4.5.2 *Wide circular tubes*

We now specialize to the case of a cylindrical tube with circular cross-section. In view of the symmetry of the problem, we expect that the height of the interface $z = u(r, \varepsilon)$ depends on the radial coordinate $r = (x^2 + y^2)^{1/2}$, and the PDE reduces to an ODE,

$$\frac{\varepsilon^2}{r}\left\{\frac{ru'}{[1+(u')^2]^{1/2}}\right\}' = u \qquad \text{in } 0 < r < 1,$$
$$u'(0) = 0, \qquad u'(1) = \tan\theta_0.$$

Here, the prime denotes a derivative with respect to r. The surface must have zero slope at $r = 0$ if it is to be smooth at the origin.

We will obtain an asymptotic solution for a wide circular tube, corresponding to the limit $\varepsilon \to 0$.* In this case, we expect that the fluid interface is almost flat over most of the interior of the tube (so that $u' \ll 1$, which linearizes the leading order equations), and rises near the boundary $r = 1$ to satisfy the boundary condition. We will obtain and match three leading-order asymptotic solutions:

(a) an inner solution valid near $r = 0$;
(b) an intermediate solution valid in $0 < r < 1$ (as we will see, this solution is the large-r limit of the inner solution, and matches with the boundary layer solution at $r = 1$);
(c) a boundary layer solution valid near $r = 1$.

Our main goal is to compute an asymptotic approximation as $\varepsilon \to 0$ for the height of the fluid at the center of the cylinder. The result is given in (4.13) below — the height is exponentially small in ε.

(a) The inner solution. We look for an inner solution near $r = 0$ of the form

$$u(r, \varepsilon) = \lambda U(R, \varepsilon), \qquad R = \frac{r}{\delta}, \tag{4.7}$$

where we will determine the scaling parameters $\lambda(\varepsilon)$, $\delta(\varepsilon)$ by matching and a dominant balance argument.

The slope u' of the interface is of the order

$$\alpha = \frac{\lambda}{\delta}.$$

Using (4.7) in the ODE, we get

$$\frac{\varepsilon^2}{\delta^2}\frac{1}{R}\left\{\frac{RU'}{[1+\alpha^2(U')^2]^{1/2}}\right\}' = U,$$

*An asymptotic solution can also be obtained for a narrow circular tube, an easier case since the problem is a regular perturbation problem as $\varepsilon \to \infty$.

where the prime denotes a derivative with respect to R. The dominant balance is $\delta = \varepsilon$, so the inner solution holds in a region of radius of the order ε about the origin.

The inner equation is then

$$\frac{1}{R}\left\{\frac{RU'}{\left[1+\alpha^2(U')^2\right]^{1/2}}\right\}' = U.$$

Since we expect that the interface is almost flat in the interior, we assume that $\alpha = o(1)$ as $\varepsilon \to 0$. (This assumption is consistent with the final solution, in which λ is exponentially small in ε.)

The leading order inner solution $U(R,\varepsilon) \sim U_0(R)$ then satisfies the linear equation

$$\begin{aligned}&\frac{1}{R}\left(RU_0'\right)' = U_0,\\ &U_0'(0) = 0.\end{aligned}$$

We do not attempt to impose the boundary condition at $r = 1$, or $R = 1/\delta$, since we do not expect the inner solution to be valid in the boundary layer where u' is of the order one.

We choose the parameter λ in (4.7) so that $U_0(0) = 1$. Thus, to leading order in ε, λ is the height of the fluid at the center of the tube. It follows that

$$U_0(R) = I_0(R),$$

where I_0 is the modified Bessel function of order zero, which satisfies [17]

$$\begin{aligned}&\frac{1}{R}\left(RI_0'\right)' - I_0 = 0,\\ &I_0(0) = 1, \quad I_0'(0) = 0.\end{aligned}$$

A power series expansion shows that there is a unique solution of this singular IVP.

The solution has the integral representation

$$I_0(R) = \frac{1}{\pi}\int_0^\pi e^{R\cos t}\,dt.$$

This function satisfies the initial conditions, and one can verify that it satisfies the ODE by direct computation and an integration by parts:

$$\begin{aligned}\frac{1}{R}\left(RI_0'\right)' - I_0 &= I_0'' - I_0 + \frac{1}{R}I_0'\\ &= \frac{1}{\pi}\int_0^\pi \left(\cos^2 t - 1\right)e^{R\cos t}\,dt + \frac{1}{R\pi}\int_0^\pi \cos t\, e^{R\cos t}\,dt\\ &= -\frac{1}{\pi}\int_0^\pi \sin^2 t\, e^{R\cos t}\,dt + \frac{1}{\pi}\int_0^\pi (\sin t)'\,\frac{e^{R\cos t}}{R}\,dt\\ &= 0.\end{aligned}$$

The asymptotic behavior of $I_0(R)$ as $R \to \infty$ can be computed from the integral representation by Laplace's method, and it grows exponentially in R. The phase $\cos t$ has a maximum at $t = 0$, and

$$\begin{aligned} I_0(R) &\sim \frac{1}{\pi}\int_0^\infty e^{R\left(1-\frac{1}{2}t^2\right)}\,dt \\ &\sim \frac{e^R}{\pi}\int_0^\infty e^{-Rt^2/2}\,dt \\ &\sim \frac{e^R}{\sqrt{2\pi R}} \end{aligned}$$

Hence, the outer expansion of this leading-order inner solution is

$$U_0(R) \sim \sqrt{\frac{e^R}{2\pi R}} \qquad \text{as } R \to \infty. \tag{4.8}$$

(b) Intermediate solution. In the region $0 < r < 1$, we expect that $u' \ll 1$. The leading order intermediate solution $u(r,\varepsilon) \sim u_0(r,\varepsilon)$ then satisfies

$$\frac{\varepsilon^2}{r}\left(ru_0'\right)' = u_0. \tag{4.9}$$

This is the same equation as the one for the inner solution, so the inner solution remains valid in this region. Nevertheless, it is instructive to obtain the asymptotic behavior of the solution directly from the ODE.

Away from $r = 0$, the solutions of (4.10) depend on two different length-scales: exponentially on a length-scale of the order ε and more slowly on a length-scale of the order one, arising from the dependence of the coefficients of the ODE on r due to the cylindrical geometry.

To account for this behavior, we use the WKB method, and look for solutions of the form

$$u_0(r,\varepsilon) = a(r,\varepsilon)e^{\varphi(r)/\varepsilon}. \tag{4.10}$$

One motivation for this form is that the constant-coefficients ODE obtained by 'freezing' the value of r at some nonzero constant value,

$$\varepsilon^2 u_0'' = u_0,$$

has solutions

$$u_0 = ae^{\pm r/\varepsilon},$$

where a is a constant. When the coefficients in the ODE depend upon r, we allow the amplitude a and the phase $\varphi(r) = \pm r$ to depend upon r in an appropriate way.

Using (4.10) in (4.9), and rewriting the result, we find that

$$a(\varphi')^2 + \varepsilon\left[2a'\varphi' + a\frac{1}{r}\left(r\varphi'\right)'\right] + \varepsilon^2\frac{1}{r}\left(ra'\right)' = a.$$

We seek an asymptotic expansion of a,

$$a(r,\varepsilon) \sim a_0(r) + \varepsilon a_1(r) + \dots \qquad \text{as } \varepsilon \to 0.$$

Using this expansion in (4.5.2), expanding, and equating coefficients of ε^0 we find that

$$a_0\left[(\varphi')^2 - 1\right] = 0.$$

Hence, if $a_0 \neq 0$, we must have

$$(\varphi')^2 = 1.$$

Omitting the constants of integration, which can be absorbed into a, the solutions are

$$\varphi(r) = \pm r.$$

Equating coefficients of ε and simplifying the result, we find that

$$a_0' + \frac{1}{2r}a_0 = 0.$$

The solution is

$$a_0(r) = \frac{A}{r^{1/2}},$$

where A is a constant.

We therefore obtain that

$$u_0(r) \sim \frac{A_+}{r^{1/2}}e^{r/\varepsilon} + \frac{A_-}{r^{1/2}}e^{-r/\varepsilon}.$$

Matching this solution as $r \to 0^+$ with the the inner solution at $r = 0$, whose outer expansion is given in (4.8), and using $R = r/\varepsilon$, $U_0 = \lambda u_0$, we find that there are no terms that grow exponentially as $r \to 0^+$ so $A_- = 0$, and

$$A_+ = \lambda\sqrt{\frac{\varepsilon}{2\pi}}.$$

Thus, the outer expansion of the inner solution (4.8) is valid as $\varepsilon \to 0$ in the interior $0 < r < 1$, and the leading order behavior of the solution is given by

$$u(r,\varepsilon) \sim \lambda\sqrt{\frac{\varepsilon}{2\pi r}}e^{r/\varepsilon} \qquad \text{as } \varepsilon \to 0. \tag{4.11}$$

Here, the height $\lambda(\varepsilon)$ of the interface at the origin remains to be determined. We will find it by matching with the solution in the boundary layer.

(c) The boundary layer solution. Since $u'(1,\varepsilon) = \tan\theta_0 > 0$, we expect that the slope u' of the interface is of the order one in a boundary layer near $r = 1$. We therefore look for an inner boundary layer solution of the form

$$u(r,\varepsilon) = \delta U(X,\varepsilon), \qquad X = \frac{1-r}{\delta}.$$

A dominant balance argument gives $\delta = \varepsilon$, and then U satisfies the ODE

$$\frac{1}{1-\varepsilon X}\left\{\frac{(1-\varepsilon X)\,U'}{[1+(U')^2]^{1/2}}\right\}' = U,$$

where the prime denotes a derivative with respect to X. The boundary conditions are

$$U'(0,\varepsilon) = -\tan\theta_0, \qquad U'(X,\varepsilon) \to 0 \quad \text{as } X \to \infty.$$

The condition as $X \to \infty$ is a matching condition, which would need to be refined for higher-order approximations.

As $\varepsilon \to 0$, we have $U(X,\varepsilon) \sim U_0(X)$ where $U_0(X)$ satisfies the following BVP

$$\begin{aligned}
&\left\{\frac{U_0'}{[1+(U_0')^2]^{1/2}}\right\}' = U_0 \qquad 0 < X < \infty,\\
&U_0'(0) = -\tan\theta_0,\\
&U_0'(X) \to 0 \quad \text{as } X \to \infty.
\end{aligned}$$

The ODE is autonomous, corresponding to a two-dimensional planar problem, and (unlike the cylindrical problem) it can be solved exactly.

To solve the equation, it is convenient to introduce the angle $\psi > 0$ of the interface, defined by

$$\tan\psi = -U_0'. \tag{4.12}$$

We will use ψ as a new independent variable, and solve for $U_0 = U_0(\psi)$ and $X = X(\psi)$. The change of variables $X \mapsto \psi$ is well-defined if U_0' is strictly decreasing, as is the case, and then $X = 0$ corresponds to $\psi = \theta_0$ and $X = \infty$ corresponds to $\psi = 0$.

Differentiating (4.12) with respect to X, and writing X-derivatives as d/dX, we find that

$$\frac{d\psi}{dX} = -\frac{d^2U_0}{dX^2}\cos^2\psi.$$

The ODE implies that

$$\frac{d^2U_0/}{dX^2} = U_0\sec^3\psi.$$

It follows that

$$\frac{d\psi}{dX} = -U_0 \sec\psi.$$

Using this result, the definition of ψ, and the equation

$$\frac{dU_0}{d\psi} = \frac{dU_0}{dX}\frac{dX}{d\psi},$$

we find that U_0, X satisfy the following ODEs:

$$\frac{dU_0}{d\psi} = \frac{\sin\psi}{U_0},$$
$$\frac{dX}{d\psi} = -\frac{\cos\psi}{U_0}.$$

The boundary conditions on $X(\psi)$ are

$$X(\theta_0) = 0, \qquad X(\psi) \to \infty \quad \text{as } \psi \to 0^+.$$

The solution for U_0 is

$$U_0(\psi) = \sqrt{2(k - \cos\psi)},$$

where k is a constant of integration. The solution for X is

$$X(\psi) = \frac{1}{\sqrt{2}} \int_\psi^{\theta_0} \frac{\cos t}{\sqrt{k - \cos t}}\, dt,$$

where we have imposed the boundary condition $X(\theta_0) = 0$. The boundary condition that $X \to \infty$ as $\psi \to 0$ implies that $k = 1$, and then

$$U_0(\psi) = 2\sin\frac{\psi}{2}, \qquad X(\psi) = \frac{1}{2}\int_\psi^{\theta_0} \frac{\cos t}{\sin\frac{t}{2}}\, dt.$$

Evaluating the integral for X, we get

$$\begin{aligned} X(\psi) &= \frac{1}{2}\int_\psi^{\theta_0} \left(\operatorname{cosec}\frac{t}{2} - \sin\frac{t}{2}\right) dt \\ &= \left[\log\tan\frac{t}{4} + 2\cos\frac{t}{2}\right]_\psi^{\theta_0} \\ &= \log\tan\frac{\theta_0}{4} + 2\cos\frac{\theta_0}{2} - \log\tan\frac{\psi}{4} - 2\cos\frac{\psi}{2}. \end{aligned}$$

The asymptotic behaviors of U_0 and X as $\psi \to 0^+$ are given by

$$U_0(\psi) = \psi + o(1),$$
$$X(\psi) = \log\tan\frac{\theta_0}{4} + 2\cos\frac{\theta_0}{2} - \log\frac{\psi}{4} - 2 + o(1).$$

It follows that the outer expansion of the leading-order boundary layer solution is

$$U_0(X) \sim 4 \tan\left(\frac{\theta_0}{4}\right) e^{-4\sin^2(\theta_0/4)} e^{-X} \qquad \text{as } X \to \infty.$$

Rewriting this expansion in terms of the original variables, $u \sim \varepsilon U_0$, $r = 1 - \varepsilon X$, we get

$$u(r, \varepsilon) \sim 4\varepsilon \tan\left(\frac{\theta_0}{4}\right) e^{-4\sin^2(\theta_0/4)} e^{-1/\varepsilon} e^{r/\varepsilon}.$$

The inner expansion as $r \to 1^-$ of the leading order intermediate solution in (4.11) is

$$u(r, \varepsilon) \sim \lambda \sqrt{\frac{\varepsilon}{2\pi}} e^{r/\varepsilon}.$$

These solutions match if

$$\lambda = 4 \tan\left(\frac{\theta_0}{4}\right) e^{-4\sin^2(\theta_0/4)} \sqrt{2\pi\varepsilon} e^{-1/\varepsilon}.$$

Thus, we conclude that

$$u(0, \varepsilon) \sim 4 \tan\left(\frac{\theta_0}{4}\right) e^{-4\sin^2(\theta_0/4)} \sqrt{2\pi\varepsilon} e^{-1/\varepsilon} \qquad \text{as } \varepsilon \to 0. \tag{4.13}$$

Chapter 5

Method of Multiple Scales: ODEs

The method of multiple scales is needed for problems in which the solutions depend simultaneously on widely different scales. A typical example is the modulation of an oscillatory solution over time-scales that are much greater than the period of the oscillations. We will begin by describing the Poincaré-Lindstedt method, which uses a 'strained' time coordinate to construct periodic solutions. We then describe the method of multiple scales.

5.1 Periodic solutions and the Poincaré-Lindstedt expansion

We begin by constructing asymptotic expansions of periodic solutions of ODEs. The first example, Duffing's equation, is a Hamiltonian system with a family of periodic solutions. The second example, van der Pol's equation, has an isolated limit cycle.

5.1.1 *Duffing's equation*

Consider an undamped nonlinear oscillator described by Duffing's equation

$$y'' + y + \varepsilon y^3 = 0,$$

where the prime denotes a derivative with respect to time t. We look for solutions $y(t, \varepsilon)$ that satisfy the initial conditions

$$y(0, \varepsilon) = 1, \qquad y'(0, \varepsilon) = 0.$$

We look for straightforward expansion of an asymptotic solution as $\varepsilon \to 0$,

$$y(t, \varepsilon) = y_0(t) + \varepsilon y_1(t) + O(\varepsilon^2).$$

The leading-order perturbation equations are

$$\begin{aligned} &y_0'' + y_0 = 0, \\ &y_0(0) = 1, \quad y_0'(0) = 0, \end{aligned}$$

with the solution

$$y_0(t) = \cos t.$$

The next-order perturbation equations are

$$\begin{aligned} &y_1'' + y_1 + y_0^3 = 0, \\ &y_1(0) = 0, \quad y_1'(0) = 0, \end{aligned}$$

with the solution

$$y_1(t) = \frac{1}{32}\left[\cos 3t - \cos t\right] - \frac{3}{8} t \sin t.$$

This solution contains a *secular term* that grows linearly in t. As a result, the expansion is not uniformly valid in t, and breaks down when $t = O(\varepsilon)$ and εy_1 is no longer a small correction to y_0.

The solution is, in fact, a periodic function of t. The straightforward expansion breaks down because it does not account for the dependence of the period of the solution on ε. The following example illustrates the difficulty.

Example 5.1 We have the following Taylor expansion as $\varepsilon \to 0$:

$$\cos\left[(1+\varepsilon)t\right] = \cos t - \varepsilon t \sin t + O(\varepsilon^2).$$

This asymptotic expansion is valid only when $t \ll 1/\varepsilon$.

To construct a uniformly valid solution, we introduced a stretched time variable

$$\tau = \omega(\varepsilon) t,$$

and write $y = y(\tau, \varepsilon)$. We require that y is a 2π-periodic function of τ. The choice of 2π here is for convenience; any other constant period — for example 1 — would lead to the same asymptotic solution. The crucial point is that the period of y in τ is independent of ε (unlike the period of y in t).

Since $d/dt = \omega d/d\tau$, the function $y(\tau, \varepsilon)$ satisfies

$$\begin{aligned} &\omega^2 y'' + y + \varepsilon y^3 = 0, \\ &y(0, \varepsilon) = 1, \quad y'(0, \varepsilon) = 0, \\ &y(\tau + 2\pi, \varepsilon) = y(\tau, \varepsilon), \end{aligned}$$

where the prime denotes a derivative with respect to τ.

We look for an asymptotic expansion of the form

$$\begin{aligned} y(\tau, \varepsilon) &= y_0(\tau) + \varepsilon y_1(\tau) + O(\varepsilon^2), \\ \omega(\varepsilon) &= \omega_0 + \varepsilon \omega_1 + O(\varepsilon^2). \end{aligned}$$

Using this expansion in the equation and equating coefficients of ε^0, we find that

$$\begin{aligned}
&\omega_0^2 y_0'' + y_0 = 0,\\
&y_0(0) = 1, \quad y_0'(0) = 0,\\
&y_0(\tau + 2\pi) = y_0(\tau).
\end{aligned}$$

The solution is

$$\begin{aligned}
&y_0(\tau) = \cos\tau,\\
&\omega_0 = 1.
\end{aligned}$$

After setting $\omega_0 = 1$, we find that the next order perturbation equations are

$$\begin{aligned}
&y_1'' + y_1 + 2\omega_1 y_0'' + y_0^3 = 0,\\
&y_1(0) = 0, \quad y_1'(0) = 0,\\
&y_1(\tau + 2\pi) = y_1(\tau).
\end{aligned}$$

Using the solution for y_0 in the ODE for y_1, we get

$$\begin{aligned}
y_1'' + y_1 &= 2\omega_1 \cos\tau - \cos^3\tau\\
&= \left(2\omega_1 - \frac{3}{4}\right)\cos\tau - \frac{1}{4}\cos 3\tau.
\end{aligned}$$

We only have a periodic solution if

$$\omega_1 = \frac{3}{8},$$

and then

$$y_1(t) = \frac{1}{32}\left[\cos 3\tau - \cos\tau\right].$$

It follows that

$$\begin{aligned}
&y = \cos\omega t + \frac{1}{32}\varepsilon\left[\cos 3\omega t - \cos\omega t\right] + O(\varepsilon^2),\\
&\omega = 1 + \frac{3}{8}\varepsilon + O(\varepsilon^2).
\end{aligned}$$

This expansion can be continued to arbitrary orders in ε.

The appearance of secular terms in the expansion is a consequence of the non-solvability of the perturbation equations for periodic solutions.

Proposition 5.2 Suppose that $f : \mathbb{T} \to \mathbb{R}$ is a smooth 2π-periodic function, where $\mathbb{T}$ is the circle of length 2π. The ODE

$$y'' + y = f$$

has a 2π-periodic solution if and only if

$$\int_{\mathbb{T}} f(t)\cos t\,dt = 0, \quad \int_{\mathbb{T}} f(t)\sin t\,dt = 0.$$

Proof. Let $L^2(\mathbb{T})$ be the Hilbert space of 2π-periodic, real-valued functions with inner product

$$\langle y, z\rangle = \int_{\mathbb{T}} y(t)z(t)\,dt.$$

We write the ODE as

$$Ay = f,$$

where

$$A = \frac{d^2}{dt^2} + 1.$$

Two integration by parts imply that

$$\begin{aligned}\langle y, Az\rangle &= \int_{\mathbb{T}} y\,(z'' + z)\;dt \\ &= \int_{\mathbb{T}} (y'' + y)\,z\,dt \\ &= \langle Ay, z\rangle,\end{aligned}$$

meaning that operator A is formally self-adjoint in $L^2(\mathbb{T})$. Hence, it follows that if $Ay = f$ and $Az = 0$, then

$$\begin{aligned}\langle f, z\rangle &= \langle Ay, z\rangle \\ &= \langle y, Az\rangle \\ &= 0.\end{aligned}$$

The null-space of A is spanned by $\cos t$ and $\sin t$. Thus, the stated condition is necessary for the existence of a solution.

When these solvability conditions hold, the method of variation of parameters can be used to construct a periodic solution

$$y(t) =$$

Thus, the conditions are also sufficient. □

In the equation for y_1, after replacing τ by t, we had

$$f(t) = 2\omega_1\cos t - \cos 3t.$$

This function is orthogonal to $\sin t$, and

$$\langle f, \cos t\rangle = 2\pi\left\{2\omega_1\overline{\cos^2 t} - \overline{\cos^4 t}\right\},$$

where the overline denotes the average value,

$$\overline{f} = \frac{1}{2\pi} \int_{\mathbb{T}} f(t)\, dt.$$

Since

$$\overline{\cos^2 t} = \frac{1}{2}, \qquad \overline{\cos^4 t} = \frac{3}{8},$$

the solvability condition implies that $\omega_1 = 3/8$.

5.1.2 *Van der Pol oscillator*

We will compute the amplitude of the limit cycle of the van der Pol equation with small damping,

$$y'' + \varepsilon \left(y^2 - 1\right) y' + y = 0.$$

This ODE describes a self-excited oscillator, whose energy increases when $|y| < 1$ and decreases when $|y| > 1$. It was proposed by van der Pol as a simple model of a beating heart. The ODE has a single stable periodic orbit, or limit cycle.

We have to determine both the period $T(\varepsilon)$ and the amplitude $a(\varepsilon)$ of the limit cycle. Since the ODE is autonomous, we can make a time-shift so that $y'(0) = 0$. Thus, we want to solve the ODE subject to the conditions that

$$\begin{aligned} &y(t + T, \varepsilon) = y(t, \varepsilon), \\ &y(0, \varepsilon) = a(\varepsilon), \\ &y'(0, \varepsilon) = 0. \end{aligned}$$

Using the Poincaré-Lindstedt method, we introduce a strained variable

$$\tau = \omega t,$$

and look for a 2π-periodic solution $y(\tau, \varepsilon)$, where $\omega = 2\pi/T$. Since $d/dt = \omega d/d\tau$, we have

$$\begin{aligned} &\omega^2 y'' + \varepsilon\omega \left(y^2 - 1\right) y' + y = 0, \\ &y(\tau + 2\pi, \varepsilon) = y(\tau, \varepsilon), \\ &y(0, \varepsilon) = a, \\ &y'(0, \varepsilon) = 0, \end{aligned}$$

where the prime denotes a derivative with respect to τ We look for asymptotic expansions,

$$\begin{aligned} y(\tau, \varepsilon) &= y_0(\tau) + \varepsilon y_1(\tau) + O(\varepsilon^2), \\ \omega(\varepsilon) &= \omega_0 + \varepsilon\omega_1 + O(\varepsilon^2), \\ a(\varepsilon) &= a_0 + \varepsilon a_1 + O(\varepsilon^2). \end{aligned}$$

Using these expansions in the equation and equating coefficients of ε^0, we find that

$$\begin{aligned}
&\omega_0^2 y_0'' + y_0 = 0, \\
&y_0(\tau + 2\pi) = y_0(\tau), \\
&y_0(0) = a_0, \\
&y_0'(0) = 0.
\end{aligned}$$

The solution is

$$\begin{aligned}
&y_0(\tau) = a_0 \cos\tau, \\
&\omega_0 = 1.
\end{aligned}$$

The next order perturbation equations are

$$\begin{aligned}
&y_1'' + y_1 + 2\omega_1 y_0'' + \left(y_0^2 - 1\right) y_0' = 0, \\
&y_1(\tau + 2\pi) = y_1(\tau), \\
&y_1(0) = a_1, \\
&y_1'(0) = 0.
\end{aligned}$$

Using the solution for y_0 in the ODE for y_1, we find that

$$y_1'' + y_1 = 2\omega_1 \cos\tau + a_0 \left(a_0^2 \cos^2\tau - 1\right) \sin\tau.$$

The solvability conditions, that the right and side is orthogonal to $\sin\tau$ and $\cos\tau$ imply that

$$\frac{1}{8} a_0^3 - \frac{1}{2} a_0 = 0, \quad \omega_1 = 0.$$

We take $a_0 = 2$; the solution $a_0 = -2$ corresponds to a phase shift in the limit cycle by π, and $a_0 = 0$ corresponds to the unstable steady solution $y = 0$. Then

$$y_1(\tau) = -\frac{1}{4} \sin 3\tau + \frac{3}{4} \sin\tau + \alpha_1 \cos\tau.$$

At the next order, in the equation for y_2, there are two free parameters, (a_1, ω_2), which can be chosen to satisfy the two solvability conditions. The expansion can be continued in the same way to all orders in ε.

5.2 The method of multiple scales

Mathieu's equation,

$$y'' + (1 + 2\varepsilon \cos 2t)\, y = 0,$$

describes a parametrically forced simple harmonic oscillator, such as a swing, whose frequency is changed slightly at twice its natural frequency.

5.3 The method of averaging

Consider a system of ODEs for $x(t) \in \mathbb{R}^n$ which can be written in the following standard form

$$x' = \varepsilon f(x, t, \varepsilon). \tag{5.1}$$

Here, $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is a smooth function that is periodic in t. We assume the period is 2π for definiteness, so that

$$f(x, t + 2\pi, \varepsilon) = f(x, t, \varepsilon).$$

Many problems can be reduced to this standard form by an appropriate change of variables.

Example 5.3 Consider a perturbed simple harmonic oscillator

$$y'' + y = \varepsilon h(y, y', \varepsilon).$$

We rewrite this equation as a first-order system and remove the unperturbed dynamics by introducing new dependent variables $x = (x_1, x_2)$ defined by

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We find, after some calculations, that (x_1, x_2) satisfy the system

$$\begin{aligned} x_1' &= -\varepsilon h\left(x_1 \cos t + x_2 \sin t, -x_1 \sin t + x_2 \cos t, \varepsilon\right) \sin t, \\ x_2' &= \varepsilon h\left(x_1 \cos t + x_2 \sin t, -x_1 \sin t + x_2 \cos t, \varepsilon\right) \cos t, \end{aligned}$$

which is in standard periodic form.

Using the method of multiple scales, we seek an asymptotic solution of (5.1) depending on a 'fast' time variable t and a 'slow' time variable $\tau = \varepsilon t$:

$$x = x(t, \varepsilon t, \varepsilon).$$

We require that $x(t, \tau, \varepsilon)$ is a 2π-periodic function of t:

$$x(t + 2\pi, \tau, \varepsilon) = x(t, \tau, \varepsilon).$$

Then $x(t, \tau, \varepsilon)$ satisfies the PDE

$$x_t + \varepsilon x_\tau = f(x, t, \varepsilon).$$

We expand

$$x(t, \tau, \varepsilon) = x_0(t, \tau) + \varepsilon x_1(t, \tau) + O(\varepsilon^2).$$

At leading order, we find that

$$x_{0t} = 0.$$

It follows that $x_0 = x_0(\tau)$ is independent of t, which is trivially a 2π-periodic function of t. At the next order, we find that x_1 satisfies

$$\begin{aligned} &x_{1t} + x_{0\tau} = f\left(x_0, t, 0\right), \\ &x_1(t + 2\pi, \tau) = x_1(t, \tau). \end{aligned} \tag{5.2}$$

The following solvability condition is immediate.

Proposition 5.4 Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is a smooth, 2π-periodic function. Then the $n \times n$ system of ODEs for $x(t) \in \mathbb{R}^n$,

$$x' = f(t),$$

has a 2π-periodc solution if and only if

$$\frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt = 0.$$

Proof. The solution is

$$x(t) = x(0) + \int_0^t f(s)\, ds.$$

We have

$$x(t + 2\pi) - x(t) = \int_t^{t+2\pi} f(s)\, ds,$$

which is zero if and only if f has zero mean over a period. □

If this condition does not hold, then the solution of the ODE grows linearly in time at a rate equal to the mean on f over a period.

An application of this proposition to (5.2) shows that we have a periodic solution for x_1 if and only if x_0 satisfies the averaged ODEs

$$x_{0\tau} = \overline{f}\left(x_0\right),$$

where

$$\overline{f}(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x, t, 0)\, dt.$$

Example 5.5 Returning to the van der Pol equation...

First we state the basic existence theorem for ODEs, which implies that the solution of (5.1) exists on a time imterval ofthe order ε^{-1}.

Theorem 5.6 Consider the IVP

$$\begin{aligned} &x' = \varepsilon f(x, t), \\ &x(0) = x_0, \end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n$ is a Lipschitz continuous function of $x \in \mathbb{R}^n$ and a continuous function of $t \in \mathbb{T}$. For $R > 0$, let

$$B_R(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < R\},$$

where $|\cdot|$ denotes the Euclidean norm,

$$|x| = \sum_{i=1}^{n} |x_i|^2 .$$

Let

$$M = \sup_{x \in B_R(x_0), t \in \mathbb{T}} |f(x,t)|.$$

Then there is a unique solution of the IVP,

$$x : (-T/\varepsilon, T/\varepsilon) \to B_R(x_0) \subset \mathbb{R}^n$$

that exists for the time interval $|t| < T/\varepsilon$, where

$$T = \frac{R}{M}.$$

Theorem 5.7 (Krylov-Bogoliubov-Mitropolski) With the same notation as the previous theorem, there exists a unique solution

$$\overline{x} : (-T/\varepsilon, T/\varepsilon) \to B_R(x_0) \subset \mathbb{R}^n$$

of the averaged equation

$$\begin{aligned} \overline{x}' &= \varepsilon \overline{f}(\overline{x}), \\ \overline{x}(0) &= x_0, \end{aligned}$$

where

$$\overline{f}(x) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x,t)\, dt.$$

Assume that $f : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n$ is continuously differentiable. Let $0 < \widetilde{R} < R$, and define

$$\widetilde{T} = \frac{\widetilde{R}}{\widetilde{M}}, \qquad \widetilde{M} = \sup_{x \in B_{\widetilde{R}}(x_0), t \in \mathbb{T}} |f(x,t)|.$$

Then there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $0 \le \varepsilon \le \varepsilon_0$

$$|x(t) - \overline{x}(t)| \le C\varepsilon \qquad \text{for } |t| \le \widetilde{T}/\varepsilon.$$

A more geometrical way to view these results is in terms of Poincaré return maps. We define the Poincaré map $P^\varepsilon(t_0) : \mathbb{R}^n \to \mathbb{R}^n$ for (5.1) as the 2π-solution map. That is, if $x(t)$ is the solution of (5.1) with the initial condition $x(t_0) = x_0$, then

$$P^\varepsilon(t_0)x_0 = x(t_0 + 2\pi).$$

The choice of t_0 is not essential here, since different choices of t_0 lead to equivalent Poincaré maps when f is a 2π-periodic function of t. Orbits of the Poincaré map consist of closely spaced points when ε is small, and they are approximated by the trajectories of the averaged equations for times $t = O(1/\varepsilon)$.

5.4 Perturbations of completely integrable Hamiltonian systems

Consider a Hamiltonian system whose configuration is described by n angles $x \in \mathbb{T}^n$ with corresponding momenta $p \in \mathbb{R}^n$. The Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ gives the energy of the system. The motion is described by Hamilton's equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}.$$

This is a $2n \times 2n$ system of ODEs for $x(t)$, $p(t)$.

Example 5.8 The simple pendulum has Hamiltonian

$$H(x, p) = \frac{1}{2}p^2 + \sin x.$$

A change of coordinates $(x, p) \mapsto (\tilde{x}, \tilde{p})$ that preserves the form of Hamilton's equations is called a canonical change of coordinates. A Hamiltonian system is completely integrable if there exists a canonical change of coordinates $(x, p) \mapsto (\varphi, I)$ such that $H = H(I)$ is independent of the angles $\varphi \in Ts^n$. In these action-angle coordinates, Hamilton's equations become

$$\frac{d\varphi}{dt} = \frac{\partial H}{\partial I}, \qquad \frac{dI}{dt} = 0.$$

Hence, the solutions are $I =$ constant and

$$\varphi(t) = \omega(I)t + \varphi_0,$$

where

$$\omega(I) = \frac{\partial H}{\partial I}.$$

5.5 The WKB method for ODEs

Suppose that the frequency of a simple harmonic oscillator is changing slowly compared with a typical period of the oscillation. For example, consider small-amplitude oscillations of a pendulum with a slowly varying length. How does the amplitude of the oscillations change?

The ODE describing the oscillator is

$$y'' + \omega^2(\varepsilon t)y = 0,$$

where $y(t, \varepsilon)$ is the amplitude of the oscillator, and $\omega(\varepsilon t) > 0$ is the slowly varying frequency.

Following the method of multiple scales, we might try to introduce a slow time variable $\tau = \varepsilon t$, and seek an asymptotic solutions

$$y = y_0(t, \tau) + \varepsilon y_1(t, \tau) + O(\varepsilon^2).$$

Then we find that

$$\begin{aligned} &y_{0tt} + \omega^2(\tau)y_0 = 0, \\ &y_0(0) = a, \quad y_0'(0) = 0, \end{aligned}$$

with solution

$$y_0(t, \tau) = a\cos\left[\omega(\tau)t\right].$$

At next order, we find that

$$y_{1tt} + \omega^2 y_1 + 2y_{0t\tau} = 0,$$

or

$$y_{1tt} + \omega^2 y_1 = 2a\omega\omega_\tau t\cos\omega t.$$

We cannot avoid secular terms that invalidate the expansion when $t = O(1/\varepsilon)$. The defect of this solution is that its period as a function of the 'fast' variable t depends on the 'slow' variable τ.

Instead, we look for a solution of the form

$$\begin{aligned} &y = y(\theta, \tau, \varepsilon), \\ &\theta = \frac{1}{\varepsilon}\varphi(\varepsilon t), \qquad \tau = \varepsilon t, \end{aligned}$$

where we require y to be 2π-periodic function of the 'fast' variable θ,

$$y(\theta + 2\pi, \tau, \varepsilon) = y(\theta, \tau, \varepsilon).$$

The choice of 2π for the period is not essential; the important requirement is that the period is a constant that does not depend upon τ.

By the chain rule, we have

$$\frac{d}{dt} = \varphi_\tau \partial_\theta + \varepsilon \partial_\tau,$$

and

$$y'' = (\varphi_\tau)^2 y_{\theta\theta} + \varepsilon \left\{ 2\varphi_\tau y_{\theta\tau} + \varphi_{\tau\tau} y_\theta \right\} + \varepsilon^2 y_{\tau\tau}.$$

It follows that y satisfies the PDE

$$(\varphi_\tau)^2 y_{\theta\theta} + \omega^2 y + \varepsilon \left\{ 2\varphi_\tau y_{\theta\tau} + \varphi_{\tau\tau} y_\theta \right\} + \varepsilon^2 y_{\tau\tau} = 0.$$

We seek an expansion

$$y(\theta, \tau, \varepsilon) = y_0(\theta, \tau) + \varepsilon y_1(\theta, \tau) + O(\varepsilon^2).$$

Then

$$(\varphi_\tau)^2 y_{0\theta\theta} + \omega^2 y_0 = 0.$$

Imposing the requirement that y_0 is a 2π-periodic function of θ, we find that

$$(\varphi_\tau)^2 = \omega^2,$$

which is satisfied if

$$\varphi(\tau) = \int_0^\tau \omega(\sigma)\, d\sigma.$$

The solution for y_0 is then

$$y_0(\theta, \tau) = A(\tau) e^{i\theta} + \text{c.c.},$$

where it is convenient to use complex exponentials, $A(\tau)$ is an arbitrary complex-valued scalar, and c.c. denotes the complex conjugate of the preceeding term.

At the next order, we find that

$$\omega^2 \left(y_{1\theta\theta} + y_1 \right) + 2\omega y_{0\theta\tau} + \omega_\tau y_{0\theta} = 0.$$

Using the solution for y_0 is this equation, we find that

$$\omega^2 \left(y_{1\theta\theta} + y_1 \right) + i \left(2\omega A_\tau + \omega_\tau A \right) e^{i\theta} + \text{c.c.} = 0.$$

The solution for y_1 is periodic in θ if and only if A satisfies

$$2\omega A_\tau + \omega_\tau A = 0.$$

It follows that

$$\left(\omega |A|^2 \right)_\tau = 0,$$

so that

$$\omega |A|^2 = \text{constant}.$$

Thus, the amplitude of the oscillator is proportional to $\omega^{-1/2}$ as its frequency changes.

The energy E ofthe oscillator is given by

$$\begin{aligned} E &= \frac{1}{2}\left(y'\right)^2 + \frac{1}{\omega^2}y^2 \\ &= \frac{1}{2}\omega^2|A|^2. \end{aligned}$$

Thus, E/ω is constant. The quantity E/ω is called the action. It is an example of an adiabatic invariant.

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