ENHANCED LIFESPAN OF SMOOTH SOLUTIONS OF A BURGERS-HILBERT EQUATION[*]

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Abstract. We consider an initial value problem for a quadratically nonlinear inviscid Burgers-Hilbert equation that models the motion of vorticity discontinuities. We use a normal form transformation, which is implemented by means of a near-identity coordinate change of the independent spatial variable, to prove the existence of small, smooth solutions over cubically nonlinear time-scales. For vorticity discontinuities, this result means that there is a cubically nonlinear time-scale before the onset of filamentation.

Key words. Normal form transformations, nonlinear waves, inviscid Burgers equation, vorticity discontinuities.

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1. Introduction. We consider the following initial value problem for an inviscid Burgers-Hilbert equation for $u(t, x; \epsilon)$:

$$\begin{aligned} u_t + \epsilon u u_x &= \mathbf{H}\left[u\right], \\ u(0, x; \epsilon) &= u_0(x). \end{aligned} \tag{1.1}$$

In (1.1), $\mathbf{H}$ is the spatial Hilbert transform, ϵ is a small parameter, and u_0 is given smooth initial data.

This Burgers-Hilbert equation is a model equation for nonlinear waves with constant frequency [1], and it provides a formal asymptotic description of the small-amplitude motion of vorticity discontinuities in two-dimensional inviscid, incompressible fluid flows [1, 8]. In particular, Biello and Hunter [1] show that the weakly nonlinear behavior of a vorticity discontinuity located at $y = \epsilon u(t, x; \epsilon)$ over cubically nonlinear time-scales of the order ϵ^{-2} is identical to that of solutions of a Burgers-Hilbert equation with suitably chosen coefficients, which may be normalized as in (1.1).

We assume for simplicity that $x \in \mathbb{R}$, in which case the Hilbert transform is given by

$$\mathbf{H}[u](t, x; \epsilon) = \text{p.v.} \frac{1}{\pi} \int \frac{u(t, y; \epsilon)}{x - y} \, dy.$$

We will show that smooth solutions of (1.1) exist for times of the order ϵ^{-2} as $\epsilon \to 0$. Explicitly, if $H^s(\mathbb{R})$ denotes the standard Sobolev space of functions with s weak L^2-derivatives, we prove the following result:

THEOREM 1.1. *Suppose that $u_0 \in H^2(\mathbb{R})$. There are constants $k > 0$ and $\epsilon_0 > 0$, depending only on $\|u_0\|_{H^2}$, such that for every ϵ with $|\epsilon| \leq \epsilon_0$ there exists a solution*

$$u \in C\left(I^\epsilon; H^2\left(\mathbb{R}\right)\right) \cap C^1\left(I^\epsilon; H^1\left(\mathbb{R}\right)\right)$$

of (1.1) defined on the time-interval $I^\epsilon = \left[-k/\epsilon^2, k/\epsilon^2\right]$.

The cubically nonlinear $O(\epsilon^{-2})$ lifespan of smooth solutions for the Burgers-Hilbert equation is longer than the quadratically nonlinear $O(\epsilon^{-1})$ lifespan for the inviscid Burgers equation $u_t + \epsilon u u_x = 0$. The explanation of this enhanced lifespan is that the quadratically nonlinear term of the order ϵ in (1.1) is nonresonant for the linearized equation. To see this, note that the solution of the linearized equation $u_t = \mathbf{H}[u]$ is given by $u = e^{t\mathbf{H}} u_0$, or

$$u(t, x) = u_0(x) \cos t + h_0(x) \sin t, \qquad h_0 = \mathbf{H}[u_0],$$

as may be verified by use of the identity $\mathbf{H}^2 = -\mathbf{I}$. This solution oscillates with frequency one between the initial data and its Hilbert transform, and the effect of the nonlinear forcing term $\epsilon u u_x$ on the linearized equation averages to zero because it contains no Fourier component in time whose frequency is equal to one. Alternatively, one can view the averaging of the nonlinearity as a consequence of the fact that the nonlinear steepening of the profile in one phase of the oscillation is canceled by its expansion in the other phase.

This phenomenon is illustrated by numerical results from [1], which are reproduced in Figure 1.1. The transition from an $O(\epsilon^{-1})$ lifespan for large ϵ to an $O(\epsilon^{-2})$ lifespan for small ϵ is remarkably rapid: once a singularity fails to form over the first oscillation in time, a smooth solution typically persists over many oscillations.
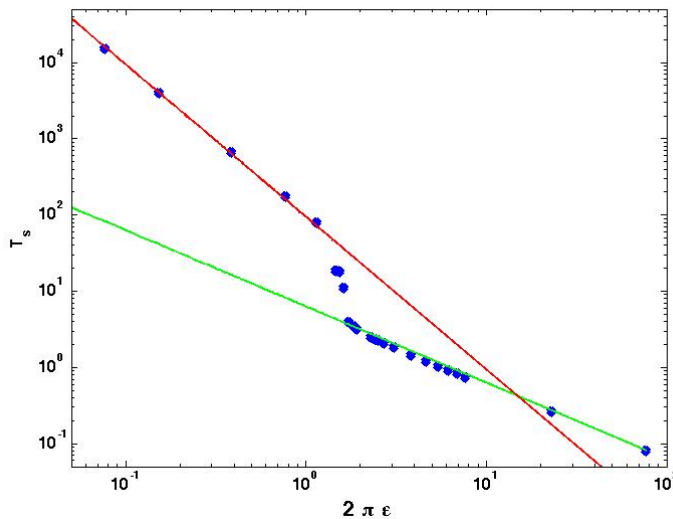

FIG. 1.1. *Logarithm of the singularity formation time T_s for the Burgers-Hilbert equation (1.1) versus the logarithm of $2\pi\epsilon$ for fixed initial data u_0. Numerical solutions are shown by diamonds. The steeper line is a formal asymptotic prediction from [1] for $\epsilon \ll 1$, which gives $T_s = 2.37\,\epsilon^{-2}$. The shallower line is the singularity formation time for the inviscid Burgers equation, which gives $T_s = \epsilon^{-1}$. (See [1] for further details.)*

In the context of the motion of a vorticity discontinuity, numerical solutions of the incompressible Euler equations for vorticity discontinuities [2, 5] show that the formation of a singularity in a solution of (1.1) (in which u_x becomes infinite) corresponds to the 'breaking' of the discontinuity. After this occurs, the discontinuity folds over and develops extremely thin, long filaments. Nevertheless, in the closely related problem of the motion of vortex patches, it has been proved that the discontinuity remains

smooth globally in time [4]. The result proved here for the Burgers-Hilbert equation therefore corresponds in the fluid problem to an enhanced $O(\epsilon^{-2})$ lifespan of a vorticity discontinuity before nonlinear 'breaking' leads to the formation of filaments.

There are three main difficulties in the proof of Theorem 1.1. The first is that the presence of a quadratically nonlinear term in (1.1) means that straightforward energy estimates prove the existence of smooth solutions only on time-scales of the order ϵ^{-1}. Following the idea introduced by Shatah [9] in the context of PDEs, and used subsequently by other authors, we remove the quadratically nonlinear term of the order ϵ by a normal form or near-identity transformation, replacing it by a cubically nonlinear term of the order ϵ^2. The second difficulty is that a standard normal form transformation of the dependent variable, of the type used by Shatah, leads to a loss of spatial derivatives because we are using a lower-order linear term $\mathbf{H}[u]$ to eliminate a higher-order nonlinear term $\epsilon u u_x$. The third difficulty is that (1.1) is nondispersive and solutions of the linearized equation oscillate but do not decay in time. As a result, we cannot use any kind of dispersive smoothing or decay to compensate for the loss of spatial derivatives. (See [10], for example, for further discussion of nonlinear dispersive waves.)

The key idea in this paper that avoids these difficulties is to make a transformation of the independent variable, rather than the dependent variable. We write

$$h(t, x; \epsilon) = \mathbf{H}[u](t, x; \epsilon) \tag{1.2}$$

and define

$$g(t, \xi; \epsilon) = h(t, x; \epsilon), \qquad x = \xi - \epsilon g(t, \xi; \epsilon). \tag{1.3}$$

Then, as we will show, the transformed function $g(t, \xi; \epsilon)$ satisfies an integro-differential equation of the form

$$
\begin{aligned}
g_t(t, \xi; \epsilon) = \text{p.v.} \frac{1}{\pi} \int \frac{g(t, \tilde{\xi}; \epsilon)}{\xi - \tilde{\xi}} \, d\tilde{\xi} \\
- \frac{1}{\pi} \epsilon^2 \partial_\xi \int (\xi - \tilde{\xi}) g_{\tilde{\xi}}(t, \tilde{\xi}; \epsilon) \phi \left(\frac{g(t, \xi; \epsilon) - g(t, \tilde{\xi}; \epsilon)}{\xi - \tilde{\xi}}; \epsilon \right) d\tilde{\xi}
\end{aligned}
\tag{1.4}
$$

where $\phi(c; \epsilon)$ is a smooth function, given in Lemma 2.1. The term of the order ϵ has been removed from (1.4), and the equation has good energy estimates that imply the enhanced lifespan of smooth solutions.

The interpretation of the transformation (1.3) is not entirely clear. On taking the Hilbert transform of (1.1) we get $h_t = -u + O(\epsilon)$, so that

$$x_t = -\epsilon g_t = -\epsilon h_t + O(\epsilon^2) = \epsilon u + O(\epsilon^2).$$

Thus the transformation $\xi \mapsto x$ in (1.3) agrees up to the order ϵ with a transformation from characteristic to spatial coordinates for (1.1). The coordinate ξ, however, differs from x even when $t = 0$, and the use of characteristic coordinates does not appear to simplify the analysis.

As a partial motivation for (1.3), we show in Section 3 that it agrees to leading order in ϵ with a standard normal form transformation of the dependent variable that is given in [1]. We were not able, however, to use the latter normal form transformation to prove Theorem 1.1 because of the loss of derivatives in the higher-order terms.

We consider (1.1) on the real line for simplicity. Equation (1.1) is nondispersive and our proof does not depend on any dispersive decay of the solutions in time, so a similar result would apply to spatially periodic solutions. Theorem 1.1 is also presumably true in H^s for any $s > 3/2$; we consider $s = 2$ to avoid complications associated with the use of fractional derivatives. A proof of singularity formation for (1.1) under certain conditions on u_0 and ϵ is given in [3].

2. Proof of the Theorem. In this section, we prove Theorem 1.1. It follows from standard energy arguments (*e.g.* [7]) that (1.1) has a unique local H^2-solution in a time-interval J^ϵ depending on the H^2-norm of the initial data and ϵ. Moreover, for any $s \geq 2$, the solution remains in H^s if the initial data is in H^s and depends continuously on the initial data in $C(J^\epsilon; H^s)$. Thus, in order to prove Theorem 1.1 it is sufficient to prove an *a priori* H^2-bound for smooth solutions $u \in C^\infty(I^\epsilon; H^\infty(\mathbb{R}))$ where $H^\infty(\mathbb{R}) = \cap_{s=1}^\infty H^s(\mathbb{R})$. To derive this bound, we first transform the equation to remove the order ϵ term and then carry out H^2-estimates on the transformed equation. The required computations, such as integrations by parts, are justified for these smooth solutions that decay at infinity.

2.1. Near-identity transformation. Let h denote the Hilbert transform of u, as in (1.2). Taking the Hilbert transform of (1.1), using the identity

$$\mathbf{H}\left[u^2 - h^2\right] = 2hu \tag{2.1}$$

and the fact that $u = -\mathbf{H}[h]$, we find that h satisfies the equation

$$h_t + \epsilon \left\{\mathbf{H}\left[hh_x\right] - h\mathbf{H}\left[h_x\right] - \mathbf{H}\left[h\right]h_x\right\} = \mathbf{H}\left[h\right]. \tag{2.2}$$

We will make the change of variables (1.3) in (2.2), so first we discuss (1.3).

The map $\xi \mapsto x$ is smoothly invertible if $|\epsilon g_\xi| < 1$, which holds by Sobolev embedding if $\|\epsilon g\|_{H^2}$ is sufficiently small. Specifically, we have the Gagliardo-Nirenberg-Moser inequality

$$\|g_\xi\|_{L^\infty} \leq N\|g\|_{L^2}^{1/4}\|g_{\xi\xi}\|_{L^2}^{3/4}, \tag{2.3}$$

where we can take, for example,

$$N = \sqrt{\frac{8}{3}}.$$

We assume throughout this section that

$$N\|\epsilon g\|_{L^2}^{1/4}\|\epsilon g_{\xi\xi}\|_{L^2}^{3/4} \leq \frac{1}{2}, \tag{2.4}$$

which ensures that

$$\|\epsilon g_\xi\|_{L^\infty} \leq \frac{1}{2}. \tag{2.5}$$

By the chain rule,

$$h_x = \frac{g_\xi}{1 - \epsilon g_\xi}, \qquad h_{xx} = \frac{g_{\xi\xi}}{(1 - \epsilon g_\xi)^3}. \tag{2.6}$$

Thus, if (2.5) holds, then

$$\int_{\mathbb{R}} h^2 \, dx = \int_{\mathbb{R}} g^2 \left(1 - \epsilon g_\xi\right) d\xi = \int_{\mathbb{R}} g^2 \, d\xi, \qquad \int_{\mathbb{R}} h_{xx}^2 \, dx = \int_{\mathbb{R}} \frac{g_{\xi\xi}^2}{\left(1 - \epsilon g_\xi\right)^5} \, d\xi.$$

Hence, since $\mathbf{H}$ is an isometry on H^s,

$$\|u\|_{L^2} = \|g\|_{L^2}, \qquad \left(\frac{2}{3}\right)^{5/2} \|g_{\xi\xi}\|_{L^2} \le \|u_{xx}\|_{L^2} \le 2^{5/2} \|g_{\xi\xi}\|_{L^2}, \qquad (2.7)$$

and H^2-estimates for g imply H^2-estimates for u.

Conversely, one can use the contraction mapping theorem on the space $C_0(\mathbb{R})$ of continuous functions that decay to zero at infinity, to show that if $h_0 \in C_0^1(\mathbb{R})$ and

$$\|\epsilon h_{0x}\|_{L^\infty} < 1 \qquad (2.8)$$

then there exists a function $g_0(\cdot; \epsilon) \in C_0(\mathbb{R})$ such that

$$h_0 \left(\xi - \epsilon g_0(\xi; \epsilon)\right) = g_0(\xi; \epsilon).$$

The function g_0 is smooth if h_0 is smooth, and $\|\epsilon g_{0\xi}\|_{L^\infty} \le 1/2$ if $\|\epsilon h_{0x}\|_{L^\infty} \le 1/3$. Thus, we can obtain initial data for g from the initial data for h.

From (1.3), we have

$$h_t = \frac{g_t}{1 - \epsilon g_\xi}, \qquad \mathbf{H}[h] = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \left[\frac{1 - \epsilon \tilde{g}_{\tilde{\xi}}}{\xi - \tilde{\xi} - \epsilon(g - \tilde{g})}\right] \tilde{g} \, d\tilde{\xi}$$

where we use the notation

$$g = g(t, \xi; \epsilon), \qquad \tilde{g} = g(t, \tilde{\xi}; \epsilon).$$

Using these expressions, together with (2.6), in (2.2) and simplifying the result, we find that $g(t, \xi; \epsilon)$ satisfies the following nonlinear integro-differential equation:

$$g_t = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{\tilde{g} + \epsilon(g - 2\tilde{g})\tilde{g}_{\tilde{\xi}} - \epsilon^2(g - \tilde{g})g_\xi \tilde{g}_{\tilde{\xi}}}{\xi - \tilde{\xi} - \epsilon(g - \tilde{g})} \, d\tilde{\xi}. \qquad (2.9)$$

We define

$$x = \xi - \epsilon g(t, \xi; \epsilon), \qquad \tilde{x} = \tilde{\xi} - \epsilon g(t, \tilde{\xi}; \epsilon),$$

as in (1.3), and rewrite the denominator in (2.9) as

$$\frac{1}{\xi - \tilde{\xi} - \epsilon(g - \tilde{g})} = \frac{1}{x - \tilde{x}} = \frac{1}{\xi - \tilde{\xi}} + \epsilon \frac{g - \tilde{g}}{(\xi - \tilde{\xi})^2} + \epsilon^2 \left(\frac{g - \tilde{g}}{\xi - \tilde{\xi}}\right)^2 \frac{1}{x - \tilde{x}}. \qquad (2.10)$$

This identity is exact and involves no approximation in ϵ. Using (2.10), we may express the integrand in (2.9), after some rearrangement, as

$$\frac{\tilde{g} + \epsilon(g - 2\tilde{g})\tilde{g}_{\tilde{\xi}} - \epsilon^2(g - \tilde{g})g_\xi \tilde{g}_{\tilde{\xi}}}{\xi - \tilde{\xi} - \epsilon(g - \tilde{g})} = \frac{\tilde{g}}{\xi - \tilde{\xi}} + \epsilon \partial_{\tilde{\xi}} \left[\frac{\tilde{g}(g - \tilde{g})}{\xi - \tilde{\xi}}\right]$$

$$+ \epsilon^2 \left\{\left(\frac{g - \tilde{g}}{\xi - \tilde{\xi}}\right)^2 \left(\frac{\tilde{g}}{x - \tilde{x}}\right) + \left(\frac{g - \tilde{g}}{\xi - \tilde{\xi}}\right) \tilde{g}_{\tilde{\xi}} \left[\frac{g - 2\tilde{g}}{\xi - \tilde{\xi}} - g_\xi\right]\right\}$$

$$+ \epsilon^3 \left(\frac{g - \tilde{g}}{\xi - \tilde{\xi}}\right)^2 \tilde{g}_{\tilde{\xi}} \left[\frac{g - 2\tilde{g}}{x - \tilde{x}} - g_\xi\right] - \epsilon^4 \left(\frac{g - \tilde{g}}{\xi - \tilde{\xi}}\right)^2 \left(\frac{g - \tilde{g}}{x - \tilde{x}}\right) g_\xi \tilde{g}_{\tilde{\xi}}.$$

The term of the order ϵ is an exact $\tilde{\xi}$-derivative, which integrates out for $g \in H^\infty(\mathbb{R})$, and the equation for g becomes

$$g_t = \frac{1}{\pi}\text{p.v.}\int_\mathbb{R} \frac{\tilde{g}}{\xi - \tilde{\xi}}\, d\tilde{\xi} + \frac{1}{\pi}\epsilon^2 \int_\mathbb{R} \left\{ \left(\frac{g - \tilde{g}}{\xi - \tilde{\xi}} \right)^2 \left(\frac{\tilde{g}}{x - \tilde{x}} \right) + \left(\frac{g - \tilde{g}}{\xi - \tilde{\xi}} \right) \tilde{g}_{\tilde{\xi}} \left[\frac{g - 2\tilde{g}}{\xi - \tilde{\xi}} - g_\xi \right] \right\} d\tilde{\xi}$$

$$+ \frac{1}{\pi}\epsilon^3 \int_\mathbb{R} \left(\frac{g - \tilde{g}}{\xi - \tilde{\xi}} \right)^2 \tilde{g}_{\tilde{\xi}} \left\{ \frac{g - 2\tilde{g}}{x - \tilde{x}} - g_\xi \left[1 + \epsilon \frac{g - \tilde{g}}{x - \tilde{x}} \right] \right\} d\tilde{\xi}.$$

Using (2.10), we may write this equation as

$$g_t = \mathbf{H}[g] + \frac{1}{\pi}\epsilon^2 \int_\mathbb{R} \left(\frac{g - \tilde{g}}{x - \tilde{x}} \right) \left\{ \left(\frac{\tilde{g}}{\xi - \tilde{\xi}} \right) \left[\frac{g - \tilde{g}}{\xi - \tilde{\xi}} - \tilde{g}_{\tilde{\xi}} \right] + \tilde{g}_{\tilde{\xi}} \left[\frac{g - \tilde{g}}{\xi - \tilde{\xi}} - g_\xi \right] \right\} d\tilde{\xi} \quad (2.11)$$

where

$$\mathbf{H}[g](t, \xi; \epsilon) = \text{p.v.}\frac{1}{\pi}\int_\mathbb{R} \frac{g(t, \tilde{\xi}; \epsilon)}{\xi - \tilde{\xi}}\, d\tilde{\xi}$$

denotes the Hilbert transform of g with respect to ξ. The integral of the order ϵ^2 in (2.11) is not a principal value integral since the integrand is a smooth function of $(\xi, \tilde{\xi})$.

Finally, we observe that this equation can be put in the form (1.4).

LEMMA 2.1. *An equivalent form of equation (2.11) is given by*

$$g_t = \mathbf{H}[g] - \frac{1}{\pi}\epsilon^2 \partial_\xi \int_\mathbb{R} (\xi - \tilde{\xi})\tilde{g}_{\tilde{\xi}} \, \phi\left(\frac{g - \tilde{g}}{\xi - \tilde{\xi}}; \epsilon \right) d\tilde{\xi}, \quad (2.12)$$

where

$$\phi(c; \epsilon) = -\frac{1}{\epsilon^2}\left\{ \log\left(1 - \epsilon c \right) + \epsilon c \right\}. \quad (2.13)$$

Proof. First, we check that (2.12) is well-defined. Abusing notation slightly, we write

$$c = \frac{g - \tilde{g}}{\xi - \tilde{\xi}}. \quad (2.14)$$

From (2.13),

$$\phi_c(c; \epsilon) = \frac{c}{1 - \epsilon c}, \quad (2.15)$$

so $|\phi(c; \epsilon)| \le c^2$ when $|\epsilon c| \le 1/2$, which is implied by (2.5). In that case

$$\left| \int_\mathbb{R} (\xi - \tilde{\xi})\tilde{g}_{\tilde{\xi}} \, \phi\left(c; \epsilon \right) d\tilde{\xi} \right| \le \int_\mathbb{R} \left| (g - \tilde{g})\, \tilde{g}_{\tilde{\xi}} c \right| d\tilde{\xi}.$$

We use $|g - \tilde{g}| \le 2\|g\|_{L^\infty}$ in the right hand side of this inequality and apply the Cauchy-Schwartz inequality to get

$$\left| \int_\mathbb{R} (\xi - \tilde{\xi})\tilde{g}_{\tilde{\xi}} \, \phi\left(c; \epsilon \right) d\tilde{\xi} \right| \le 2\|g\|_{L^\infty}\|g_\xi\|_{L^2}\|c\|_{L^2} \quad (2.16)$$

where

$$\|c\|_{L^2} = \left[\int_{\mathbb{R}} \left(\frac{g - \tilde{g}}{\xi - \tilde{\xi}} \right)^2 d\tilde{\xi} \right]^{1/2}$$

denotes the L^2-norm of c with respect to $\tilde{\xi}$, which is a function of ξ. Temporarily suppressing the $(t; \epsilon)$-variables and denoting the derivative of g with respect to ξ by $g'(\xi) = g_\xi(\xi)$, we have from the Taylor integral formula that

$$c = \int_0^1 g' \left(\xi + r(\tilde{\xi} - \xi) \right) dr,$$

and the Cauchy-Schwartz inequality implies that

$$\|c\|_{L^2}^2 (\xi) = \int_{\mathbb{R}} c^2 \, d\tilde{\xi}$$
$$= \int_0^1 \int_0^1 \int_{\mathbb{R}} g' \left(\xi + r(\tilde{\xi} - \xi) \right) g' \left(\xi + s(\tilde{\xi} - \xi) \right) d\tilde{\xi} dr ds$$
$$\leq \int_0^1 \int_0^1 \left(\int_{\mathbb{R}} g'^2 \left(\xi + r(\tilde{\xi} - \xi) \right) d\tilde{\xi} \right)^{1/2} \left(\int_{\mathbb{R}} g'^2 \left(\xi + s(\tilde{\xi} - \xi) \right) d\tilde{\xi} \right)^{1/2} dr ds$$
$$\leq \left(\int_0^1 \int_0^1 \frac{1}{\sqrt{rs}} dr ds \right) \left(\int_{\mathbb{R}} g'^2(\xi) \, d\xi \right)$$
$$\leq 4 \left(\int_{\mathbb{R}} g'^2(\xi) \, d\xi \right).$$

Thus,

$$\sup_{\xi \in \mathbb{R}} \left(\int_{\mathbb{R}} c^2 \, d\tilde{\xi} \right)^{1/2} \leq 2 \|g_\xi\|_{L^2}. \tag{2.17}$$

Using this estimate in (2.16), we get

$$\sup_{\xi \in \mathbb{R}} \left| \int_{\mathbb{R}} (\xi - \tilde{\xi}) \tilde{g}_{\tilde{\xi}} \, \phi(c; \epsilon) \, d\tilde{\xi} \right| \leq 4 \|g\|_{L^\infty} \|g_\xi\|_{L^2}^2.$$

Thus, the $\tilde{\xi}$-integral in (2.12) converges when $g \in H^1(\mathbb{R})$ and is, in fact, a uniformly bounded function of ξ.

To verify that (2.12) agrees with (2.11), we take the ξ-derivative under the integral in (2.12), use (2.15) which implies that

$$\phi_c(c; \epsilon) = \frac{g - \tilde{g}}{x - \tilde{x}},$$

and integrate by parts in the result. This gives

$$g_t = \mathbf{H}[g] + \frac{1}{\pi} \epsilon^2 \int_{\mathbb{R}} \left(\frac{g - \tilde{g}}{x - \tilde{x}} \right) \left[\tilde{g} c_{\tilde{\xi}} - (\xi - \tilde{\xi}) \tilde{g}_{\tilde{\xi}} c_\xi \right] d\tilde{\xi}. \tag{2.18}$$

Using the equations

$$c_{\tilde{\xi}} = \frac{c - \tilde{g}_{\tilde{\xi}}}{\xi - \tilde{\xi}}, \qquad c_\xi = -\frac{c - g_\xi}{\xi - \tilde{\xi}}, \tag{2.19}$$

in (2.18) and comparing the result with (2.11) proves the lemma. $\square$

2.2. Energy Estimates. Multiplying (2.12) by g, integrating the result with respect to ξ, and integrating by parts with respect to ξ, we find that the right-hand side vanishes by skew-symmetry in $(\xi, \tilde{\xi})$ so that

$$\frac{d}{dt} \|g\|_{L^2} = 0. \tag{2.20}$$

The conservation of $\|g\|_{L^2}$ is consistent with the conservation of $\|u\|_{L^2}$, which follows from (1.1). Hence, from (2.7), we have

$$\|g\|_{L^2} = \|g_0\|_{L^2} = \|u_0\|_{L^2}.$$

We also expect to be able to obtain an estimate for $\|g_{\xi\xi}\|_{L^2}$ since we can estimate $\|u_{xx}\|_{L^2}$ from (1.1), and we show explicitly that this is the case.

Differentiating (2.12) twice with respect to ξ, multiplying the result by $g_{\xi\xi}$, integrating with respect to ξ, and integrating by parts with respect to ξ, we get

$$\frac{d}{dt} \int_{\mathbb{R}} g_{\xi\xi}^2 \, d\xi = \frac{\epsilon^2}{\pi} I \tag{2.21}$$

where

$$I = \int_{\mathbb{R}^2} g_{\xi\xi\xi} \partial_\xi^2 \left[(\xi - \tilde{\xi}) \tilde{g}_{\tilde{\xi}} \phi(c; \epsilon) \right] \, d\xi d\tilde{\xi}. \tag{2.22}$$

The following lemma estimates I in terms of the H^2-norm of g.

LEMMA 2.2. *Suppose that I is given by (2.22) where ϕ is defined in (2.13), and c is defined in (2.14). There exists a numerical constant $A > 0$ such that*

$$|I| \leq A \|g_\xi\|_{L^2} \|g_{\xi\xi}\|_{L^2}^3 \tag{2.23}$$

whenever $g \in H^\infty(\mathbb{R})$ satisfies (2.4).

Proof. We first convert the $\tilde{\xi}$-derivative in the expression (2.22) for I to a ξ-derivative. Let

$$\Phi'(c; \epsilon) = \phi(c; \epsilon),$$

where a prime on Φ and related functions denotes a derivative with respect to c. It follows from (2.19) that

$$(\xi - \tilde{\xi}) \tilde{g}_{\tilde{\xi}} \phi(c; \epsilon) = (\xi - \tilde{\xi}) \left[c - (\xi - \tilde{\xi}) c_{\tilde{\xi}} \right] \phi(c; \epsilon)$$
$$= (\xi - \tilde{\xi}) c \phi(c; \epsilon) - (\xi - \tilde{\xi})^2 \Phi_{\tilde{\xi}}(c; \epsilon).$$

We use this equation in (2.22) and integrate by parts with respect to $\tilde{\xi}$ in the term involving Φ. Since g is independent of $\tilde{\xi}$, this gives

$$I = \int_{\mathbb{R}^2} g_{\xi\xi\xi} \partial_\xi^2 \left[(\xi - \tilde{\xi}) \Psi(c; \epsilon) \right] \, d\xi d\tilde{\xi} \tag{2.24}$$

where

$$\Psi(c; \epsilon) = c \phi(c; \epsilon) - 2\Phi(c; \epsilon).$$

Expanding the derivatives with respect to ξ in (2.24), we have

$$I = \int_{\mathbb{R}^2} g_{\xi\xi\xi} \left\{ (\xi - \tilde{\xi}) \left[\Psi'(c; \epsilon)c_{\xi\xi} + \Psi''(c; \epsilon)c_\xi^2 \right] + 2\Psi'(c; \epsilon)c_\xi \right\} d\xi d\tilde{\xi}.$$

Using (2.14) to express $c_{\xi\xi}$ in terms of $g_{\xi\xi}$, we get

$$c_{\xi\xi} = \frac{g_{\xi\xi} - 2c_\xi}{\xi - \tilde{\xi}}. \tag{2.25}$$

It follows that

$$I = \int_{\mathbb{R}^2} g_{\xi\xi\xi} \left[g_{\xi\xi} \Psi'(c; \epsilon) + (\xi - \tilde{\xi})\Psi''(c; \epsilon)c_\xi^2 \right] d\xi d\tilde{\xi}$$

$$= \int_{\mathbb{R}^2} \left[\frac{1}{2}\partial_\xi (g_{\xi\xi})^2 \, \Psi'(c; \epsilon) + g_{\xi\xi\xi}(\xi - \tilde{\xi})\Psi''(c; \epsilon)c_\xi^2 \right] d\xi d\tilde{\xi}.$$

Suppressing the argument of the function Ψ, and integrating by parts with respect to ξ in the result to remove the third-order derivative of g, we find that I can be expressed as

$$I = -\int_{\mathbb{R}^2} \left\{ \frac{1}{2}\Psi'' c_\xi g_{\xi\xi}^2 + g_{\xi\xi} \left[\Psi'' c_\xi^2 + (\xi - \tilde{\xi})\Psi''' c_\xi^3 + 2(\xi - \tilde{\xi})\Psi'' c_\xi c_{\xi\xi} \right] \right\} d\xi d\tilde{\xi}.$$

Using (2.25) in this equation to replace $c_{\xi\xi}$ by $g_{\xi\xi}$, we get

$$I = -\frac{5}{2}I_1 + 3I_2 - I_3$$

where

$$I_1 = \int_{\mathbb{R}^2} \Psi''(c; \epsilon)c_\xi g_{\xi\xi}^2 \, d\xi d\tilde{\xi},$$

$$I_2 = \int_{\mathbb{R}^2} \Psi''(c; \epsilon)c_\xi^2 g_{\xi\xi} \, d\xi d\tilde{\xi}, \tag{2.26}$$

$$I_3 = \int_{\mathbb{R}^2} (\xi - \tilde{\xi})\Psi'''(c; \epsilon)c_\xi^3 g_{\xi\xi} \, d\xi d\tilde{\xi}.$$

The functions Ψ'', Ψ''' are given explicitly by

$$\Psi''(c; \epsilon) = \frac{c}{(1 - \epsilon c)^2}, \qquad \Psi'''(c; \epsilon) = \frac{1 + \epsilon c}{(1 - \epsilon c)^3}.$$

In particular, if $|\epsilon c| \le 1/2$, which is the case if g satisfies (2.4), then

$$|\Psi''(c; \epsilon)| \le 4|c|, \qquad |\Psi'''(c; \epsilon)| \le 12. \tag{2.27}$$

We will estimate the terms in (2.26) separately.

Estimating I_1: Using (2.27) in (2.26), we get that

$$|I_1| \le 4 \int_{\mathbb{R}^2} \left| cc_\xi g_{\xi\xi}^2 \right| d\tilde{\xi} d\xi$$

$$\le 4 \sup_{\xi \in \mathbb{R}} \left[\int_{\mathbb{R}} |cc_\xi| \, d\tilde{\xi} \right] \left(\int_{\mathbb{R}} g_{\xi\xi}^2 \, d\xi \right) \tag{2.28}$$

$$\le 4 \sup_{\xi \in \mathbb{R}} \left[\left(\int_{\mathbb{R}} c^2 \, d\tilde{\xi} \right)^{1/2} \left(\int_{\mathbb{R}} c_\xi^2 \, d\tilde{\xi} \right)^{1/2} \right] \left(\int_{\mathbb{R}} g_{\xi\xi}^2 \, d\xi \right).$$

By a similar argument to the proof of (2.17), using Taylor's theorem with integral remainder and the Cauchy-Schwartz inequality, we have from (2.14) and (2.19) that

$$
\int_{\mathbb{R}} c_{\tilde{\xi}}^2 \, d\tilde{\xi} = \int_{\mathbb{R}} \left[\frac{\tilde{g} - g - (\tilde{\xi} - \xi)g_\xi}{(\tilde{\xi} - \xi)^2} \right]^2 \, d\tilde{\xi}
$$
$$
= \int_0^1 \int_0^1 \int_{\mathbb{R}} (1-r)(1-s) g'' \left(\xi + r(\tilde{\xi} - \xi) \right) g'' \left(\xi + s(\tilde{\xi} - \xi) \right) \, d\tilde{\xi} dr ds
$$
$$
\leq \int_0^1 \int_0^1 (1-r)(1-s)
$$
$$
\left(\int_{\mathbb{R}} g''^2 \left(\xi + r(\tilde{\xi} - \xi) \right) \, d\tilde{\xi} \right)^{1/2} \left(\int_{\mathbb{R}} g''^2 \left(\xi + s(\tilde{\xi} - \xi) \right) \, d\tilde{\xi} \right)^{1/2} \, dr ds
$$
$$
\leq \left(\int_0^1 \int_0^1 \frac{(1-r)(1-s)}{\sqrt{rs}} \, dr ds \right) \left(\int_{\mathbb{R}} g_{\xi\xi}^2 (\xi) \, d\xi \right)
$$
$$
\leq \frac{16}{9} \| g_{\xi\xi} \|_{L^2}^2.
$$

Thus,

$$
\sup_{\xi \in \mathbb{R}} \left(\int_{\mathbb{R}} c_{\tilde{\xi}}^2 \, d\tilde{\xi} \right)^{1/2} \leq \frac{4}{3} \| g_{\xi\xi} \|_{L^2}. \tag{2.29}
$$

Using (2.17) and (2.29) in (2.28), we get that

$$
|I_1| \leq A_1 \| g_\xi \|_{L^2} \| g_{\xi\xi} \|_{L^2}^3,
$$

where $A_1 = 32/3$ is a numerical constant.

Estimating I_2: Using (2.27) and (2.29) in (2.26), we get that

$$
|I_2| \leq 4 \int_{\mathbb{R}^2} \left| c c_{\tilde{\xi}}^2 g_{\xi\xi} \right| \, d\tilde{\xi} d\xi
$$
$$
\leq 4 \int_{\mathbb{R}} \left(\sup_{\tilde{\xi} \in \mathbb{R}} |c| \right) \left(\int_{\mathbb{R}} c_{\tilde{\xi}}^2 \, d\tilde{\xi} \right) |g_{\xi\xi}| \, d\xi \tag{2.30}
$$
$$
\leq \frac{64}{9} \| g_{\xi\xi} \|_{L^2}^2 \int_{\mathbb{R}} \left(\sup_{\tilde{\xi} \in \mathbb{R}} |c| \right) |g_{\xi\xi}| \, d\xi.
$$

Suppressing the $(t; \epsilon)$-variables, we observe from (2.14) that

$$
\sup_{\tilde{\xi} \in \mathbb{R}} |c| = \sup_{\tilde{\xi} \in \mathbb{R}} \left| \frac{g - \tilde{g}}{\xi - \tilde{\xi}} \right|
$$
$$
= \sup_{\tilde{\xi} \in \mathbb{R}} \left| \frac{1}{\xi - \tilde{\xi}} \int_{\tilde{\xi}}^{\xi} g'(z) \, dz \right|
$$
$$
\leq g_\xi^*(\xi),
$$

where

$$
g_\xi^*(\xi) = \sup_{\tilde{\xi} \in \mathbb{R}} \frac{1}{|\xi - \tilde{\xi}|} \left| \int_{\tilde{\xi}}^{\xi} |g'(z)| \, dz \right|
$$

is the maximal function of $g' = g_\xi$, defined using intervals whose left or right endpoint is ξ.

Using this inequality and the Cauchy-Schwartz inequality in (2.30), we find that

$$|I_2| \leq \frac{64}{9}\|g_\xi^*\|_{L^2}\|g_{\xi\xi}\|_{L^2}^3.$$

The maximal operator is bounded on L^2, so there exists a numerical constant M such that

$$\|g_\xi^*\|_{L^2} \leq M\|g_\xi\|_{L^2}. \tag{2.31}$$

For example, from [6], we can take

$$M = 1 + \sqrt{2}.$$

It follows that

$$|I_2| \leq A_2\|g_\xi\|_{L^2}\|g_{\xi\xi}\|_{L^2}^3$$

where $A_2 = 64M/9$.

Estimating I_3: Using (2.19) in (2.26), we we can rewrite I_3 as

$$I_3 = \int_{\mathbb{R}^2} \Psi'''(c;\epsilon)g_{\xi\xi}(c - g_\xi)c_\xi^2 \, d\xi d\tilde{\xi}.$$

Splitting this integral into two terms, we get $I_3 = I_3' - I_3''$ where

$$I_3' = \int_{\mathbb{R}^2} \Psi'''(c;\epsilon)cc_\xi^2 g_{\xi\xi} \, d\xi d\tilde{\xi}, \qquad I_3'' = \int_{\mathbb{R}^2} \Psi'''(c;\epsilon)c_\xi^2 g_\xi g_{\xi\xi} \, d\xi d\tilde{\xi}.$$

Using (2.27), we have

$$|I_3'| \leq 12 \int_{\mathbb{R}^2} |cc_\xi^2 g_{\xi\xi}| \, d\xi d\tilde{\xi}, \qquad |I_3''| \leq 12 \int_{\mathbb{R}^2} |c_\xi^2 g_\xi g_{\xi\xi}| \, d\xi d\tilde{\xi}.$$

We estimate I_3' in exactly the same way as I_2, which gives

$$|I_3'| \leq A_3'\|g_\xi\|_{L^2}\|g_{\xi\xi}\|_{L^2}^3$$

where $A_3' = 64M/3$. We estimate I_3'' in a similar way to I_1 as

$$|I_3''| \leq 12 \sup_{\xi \in \mathbb{R}} \left[\int_{\mathbb{R}} c_\xi^2 \, d\tilde{\xi}\right] \left(\int_{\mathbb{R}} |g_\xi g_{\xi\xi}| \, d\xi\right),$$

which by use of (2.29) and the Cauchy-Schwartz inequality gives

$$|I_3''| \leq A_3''\|g_\xi\|_{L^2}\|g_{\xi\xi}\|_{L^2}^3$$

where $A_3'' = 64/3$.

Combining these estimates, we get (2.23) with

$$A = 48 + \frac{128}{3}M \tag{2.32}$$

where M is the maximal-function constant in (2.31). $\square$

Using (2.23) in (2.21), we find that

$$\frac{d}{dt}\|g_{\xi\xi}\|_{L^2} \leq \frac{1}{2\pi}\epsilon^2 A\|g_\xi\|_{L^2}\|g_{\xi\xi}\|_{L^2}^2.$$

Since $\|g_\xi\|_{L^2}^2 \leq \|g\|_{L^2}\|g_{\xi\xi}\|_{L^2}$ and $\|g\|_{L^2} = \|g_0\|_{L^2}$ is conserved, we get

$$\frac{d}{dt}\|g_{\xi\xi}\|_{L^2} \leq \frac{1}{2\pi}\epsilon^2 A\|g_0\|_{L^2}^{1/2}\|g_{\xi\xi}\|_{L^2}^{5/2} \tag{2.33}$$

provided that (2.4) holds. It follows from (2.33) and Gronwall's inequality that if $|\epsilon| \leq \epsilon_0$, where ϵ_0 is sufficiently small, then $\|g_{\xi\xi}\|_{L^2}$ remains finite and (2.4) holds in some time-interval $0 \leq t \leq k/\epsilon^2$, where the constants $\epsilon_0, k > 0$ may be chosen to depend only on $\|u_0\|_H^2$. The same estimates hold backward in time, so this completes the proof of Theorem 1.1.

By solving the differential inequality (2.33) subject to the constraint (2.4), we can obtain explicit expressions for ϵ_0 and k. Let

$$E_0 = \|g_0\|_{L^2}^{1/4}\|g_{0\xi\xi}\|_{L^2}^{3/4},$$

which is comparable to $\|u_0\|_{H^2}$ from (2.7). Then we find that Theorem 1.1 holds with

$$\epsilon_0 = \frac{1}{2\sqrt{2}N}\frac{1}{E_0}, \qquad k = \frac{2\pi}{3A}\frac{1}{E_0^2}$$

where N is the constant in (2.3) and A is the constant in (2.32).

3. Normal form transformation. In this section, we relate the near-identity transformation of the independent variables used above to a standard normal form transformation of the dependent variables of the form [9]

$$v = u + B(u, u)$$

where B is a bilinear form.

We consider the normal form transformation $u \mapsto v$ given in [1]:

$$v = u + \frac{1}{2}\epsilon|\partial_x|(h^2), \qquad h = \mathbf{H}[u]. \tag{3.1}$$

Here, ∂_x denotes the derivative with respect to x and $|\partial_x| = \mathbf{H}\partial_x$. Differentiating (3.1) with respect to t, using (1.1) to eliminate u_t, and simplifying the result by means of the Hilbert transform identity (2.1), we find that this transformation removes the nonresonant term of the order ϵ from the equation and gives

$$v_t + \frac{1}{2}\epsilon^2|\partial_x|\left[h|\partial_x|(u^2)\right] = \mathbf{H}[v]. \tag{3.2}$$

The bilinear form in (3.1)

$$B(u, u) = \frac{1}{2}\epsilon|\partial_x|(h^2)$$

is not bounded on H^2, but one can show that the normal form transformation (3.1) is invertible on a bounded set in H^2 when ϵ is sufficiently small. We omit the proof, since this invertibility does not appear to be useful here.

The main difficulty with the standard normal form transformation (3.1) is that we cannot obtain good H^2-estimates for v from the transformed equation (3.2). Equation (3.2) contains second-order derivatives, rather than first-order derivatives as in (1.1), and there is a loss of derivatives. Even knowing that we can estimate the H^s-norm of u in terms of the H^s-norm of v, we need to use the H^3-norm of v to estimate the time-derivative of the H^2-norm of v. Moreover, the use of the original Burgers-Hilbert equation (1.1) to estimate the growth of higher-derivative norms of u, and therefore v, in terms of the H^2-norm of u appears to lead only to a logarithmic improvement in the life-span of smooth solutions over the $O(\epsilon^{-1})$ life-span that follows from straightforward energy estimates.

The appearance of additional derivatives in the transformed equation is a consequence of using a zeroth-order linear term $\mathbf{H}[u]$ to remove a first-order quadratic term $\epsilon u u_x$. In fact, for every power of ϵu that one gains through a normal form transformation of the dependent variable, one introduces an additional derivative.

By contrast, higher-order linear terms lead to normal form transformations that are easier to analyze. For example, consider the KdV equation

$$u_t + \epsilon u u_x = u_{xxx}.$$

Then, assuming we can ignore difficulties associated with low wavenumbers (*e.g.* by considering spatially periodic solutions with zero mean), we find that the normal form transformation

$$v = u - \frac{1}{6}\epsilon \left(\partial_x^{-1} u\right)^2$$

leads to the equation

$$v_t - \frac{1}{6}\epsilon^2 u^2 \left(\partial_x^{-1} u\right) = v_{xxx}.$$

In this case, the normal form transformation is bounded and it smooths the nonlinear term.

To explain the connection between the normal form transformation (3.1) and the near-identity transformation (1.3), we reformulate (3.1) as a backward-Euler approximation of an evolution equation in ϵ. Writing $g = \mathbf{H}[v]$ and taking the Hilbert transform of (3.1), we get the ODE

$$g = h - \epsilon h h_x. \tag{3.3}$$

We regard $g(t, x)$ as a given function and use (3.3) to determine the corresponding function h. We may write (3.3) as

$$\frac{h - g}{\epsilon} - h h_x = 0,$$

which agrees up to the order ϵ with an evolution equation in ϵ for $h(t, x; \epsilon)$:

$$h_\epsilon - h h_x = 0, \qquad h(t, x; 0) = g(t, x). \tag{3.4}$$

By the method of characteristics, the solution of (3.4) is

$$h(t, x; \epsilon) = g(t, \xi), \qquad x = \xi - \epsilon g(t, \xi),$$

which is the transformation (1.3). Since (1.3) agrees to the order ϵ with a normal form transformation that removes the order ϵ term from (1.1), this transformation must do so also, as we verified explicitly in Section 2.

It is rather remarkable that the normal form transformation (3.1) can be implemented by making a change of spatial coordinate in the equation for h, but we do not have a good explanation for why this should be possible. The idea of replacing an unbounded normal form transformation, such as (3.1), by a bounded normal form flow, such as (3.4), may also be useful in other problems.

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