

Diffractive Nonlinear Geometrical Optics for Variational Wave Equations and the Einstein Equations

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Abstract

We derive an asymptotic solution of the vacuum Einstein equations that describes the propagation and diffraction of a localized, large-amplitude, rapidly-varying gravitational wave. We compare and contrast the resulting theory of strongly nonlinear geometrical optics for the Einstein equations with nonlinear geometrical optics theories for variational wave equations. © 2000 Wiley Periodicals, Inc.

1 Introduction

Geometrical optics¹ and its generalizations, such as the geometrical theory of diffraction, are a powerful approach to the study of wave propagation, for both linear and nonlinear waves. In this paper, we develop a theory of strongly nonlinear geometrical optics for gravitational wave solutions of the vacuum Einstein equations. Specifically, we derive asymptotic equations that describe the diffraction of localized, large-amplitude, rapidly-varying gravitational waves.

These equations are a generalization of the straightforward non-diffractive, nonlinear geometrical optics equations for large-amplitude gravitational waves derived in [2]. This strongly nonlinear theory differs fundamentally from the weakly nonlinear theory for small-amplitude gravitational waves obtained by Choquet-Bruhat [11] and Isaacson [22] because it captures the direct nonlinear self-interaction of the waves.

¹Here, we use the term ‘geometrical optics’ to refer to any asymptotic theory for the propagation of short-wavelength, high-frequency waves, irrespective of its area of physical application.

The Einstein equations may be derived from a variational principle, and — when written with respect to a suitable gauge — they form a system of wave equations for the gravitational field. In order to explain the structure of nonlinear geometrical optics theories for gravitational waves and to motivate the form of our asymptotic expansion, it is useful to consider first such theories for a general class of variational wave equations.

We describe straightforward and diffractive geometrical optics theories for variational wave equations in Section 2. In the weakly nonlinear theory for waves with periodic waveforms, the scalar amplitude-waveform function $a(\theta, v)$ of the wave depends on a ray variable v and periodically on a ‘fast’ phase variable θ . The wave-amplitude a satisfies the Hunter-Saxton equation (2.12),

$$(1.1) \quad \left\{ a_v + \left(\frac{1}{2} \Lambda a^2 \right)_\theta + N a \right\}_\theta = \frac{1}{2} \Lambda \{ a_\theta^2 - \langle a_\theta^2 \rangle \},$$

where the angular brackets denote an average with respect to θ . The coefficient Λ of the nonlinear terms in (1.1) may be interpreted as a derivative of the wave speed with respect to the wave amplitude. In addition, the wave-amplitude is coupled with a slowly-varying mean field. The mean-field satisfies a system (2.22) of variational wave equations of the same form as the original system with a source term proportional to the mean energy-density of the wave $\langle a_\theta^2 \rangle$.

An application of this weakly-nonlinear expansion to the Einstein equations leads to the theory of Choquet-Bruhat [11] and Isaacson [22]. In that case, the nonlinear coefficients corresponding to Λ in the equations for the amplitudes of the gravitational waves are identically zero. This theory therefore describes the nonlinear interaction between a high-frequency, oscillatory gravitational wave and a slowly-varying mean gravitational field, but does not describe the direct nonlinear self-interaction of the gravitational wave itself.

In Section 2.4, we make a distinction between ‘genuinely nonlinear’ wave-fields in variational wave equations, for which Λ is never zero, and ‘linearly degenerate’ wave-fields for which Λ is identically zero (see Definition 2.1). All wave-fields in the Einstein equations are linearly degenerate. As observed in [13, 14, 16, 19, 29], for example, this fact reflects a fundamental degeneracy in the nonlinearity of the Einstein equations in comparison with general quasilinear wave equations.

A property of the Einstein equations related to their linear degeneracy is that they possess an exact solution for non-distorting, large-amplitude, plane waves, the Brinkmann solution [9, 10, 26, 27]. One can use this solution as the basis of a strongly nonlinear geometrical optics theory for large-amplitude gravitational waves, which leads to a generalization of the colliding plane wave equations [2]. We outline the resulting non-diffractive theory in Section 2.5.

We derive our diffractive, strongly nonlinear geometrical optics solution of the vacuum Einstein equations in Section 3. This solution describes the propagation and diffraction of a thin, large-amplitude gravitational wave, such as a pulse or ‘sandwich’ wave. The simplest, and most basic, case is that of a plane-polarized

gravitational wave diffracting in a single direction. We summarize the resulting asymptotic equations here.

We suppose that the polarization of the wave is aligned with the diffraction direction. Then, with respect to a suitable coordinate system (u, v, y, z) , the metric \mathbf{g} of the wave adopts the form

$$(1.2) \quad \mathbf{g} = -2e^{-M}(du - \varepsilon Y dy)dv + e^{-U}(e^V dy^2 + e^{-V} dz^2) + O(\varepsilon^2).$$

Here, ε is a small parameter. The leading-order metric component functions U , V , M and the first-order function Y depend upon a ‘fast’ phase variable θ , an ‘intermediate’ transverse variable η , and the ‘slow’ ray variable v , where

$$\theta = \frac{u}{\varepsilon^2}, \quad \eta = \frac{y}{\varepsilon}.$$

The phase u is light-like and the transverse coordinate y is space-like.

This metric describes a gravitational wave whose wavefronts are close to the null-hypersurface $u = 0$. The wave is plane-polarized in the (y, z) -directions and diffracts in the y -direction.

To write equations for the metric component functions in a concise form, we define a derivative D_η and functions ϕ, ψ by

$$(1.3) \quad D_\eta = e^U (\partial_\eta + Y \partial_\theta),$$

$$(1.4) \quad \phi = D_\eta M - e^U Y_\theta,$$

$$(1.5) \quad \psi = D_\eta(U + V).$$

We note that $\varepsilon D_\eta = e^V (dy)^\sharp$, where \sharp denotes the ‘raising’ operator from one-forms to vector fields.

Then (U, V, M, Y) are functions of (θ, η, v) which satisfy the following system of PDEs:

$$(1.6) \quad U_{\theta\theta} - \frac{1}{2}(U_\theta^2 + V_\theta^2) + U_\theta M_\theta = 0,$$

$$(1.7) \quad (\phi + \psi)_\theta = \psi(U + V)_\theta,$$

$$(1.8) \quad U_{\theta v} - U_\theta U_v = \frac{1}{2}e^{-(U+V+M)} \left\{ D_\eta \phi + D_\eta \psi - \frac{1}{2}\phi^2 - \phi\psi - \psi^2 \right\},$$

$$(1.9) \quad V_{\theta v} - \frac{1}{2}(U_\theta V_v + U_v V_\theta) = \frac{1}{2}e^{-(U+V+M)} \left\{ -D_\eta \phi + \frac{1}{2}\phi^2 \right\},$$

$$(1.10) \quad M_{\theta v} + \frac{1}{2}(U_\theta U_v - V_\theta V_v) = \frac{1}{2}e^{-(U+V+M)} \left\{ -D_\eta \psi - \frac{1}{2}\phi^2 + \psi^2 \right\}.$$

This system is the main result of our analysis. It is an asymptotic reduction of the full vacuum Einstein equations to a $(1 + 2)$ -dimensional system of PDEs. The system provides a model nonlinear wave equation for general relativity, and should be useful, for example, in studying the focusing of nonplanar gravitational waves and the effect of diffraction on the formation of singularities. We plan to study these topics in future work.

We discuss (1.6)–(1.10) in Section 4. The constraint equation (1.6) is a nonlinear ODE in θ in which η, v appear as parameters. As shown in Proposition 4.1, this constraint is preserved by the remaining equations. Equation (1.7) is a linear, non-homogeneous ODE in θ for Y , whose coefficients depend on θ - and η -derivatives of (U, V, M) . It may therefore be regarded as determining Y in terms of (U, V, M) . Equations (1.8)–(1.10) form a system of evolution equations for (U, V, M) which is coupled with Y . The main part of the system consists of a (1+2)-dimensional wave equation in (θ, η, v) for $(U + V)$, where (θ, v) are characteristic coordinates.

When $Y = 0$ and all functions are independent of η , equations (1.6)–(1.10) reduce to the colliding plane wave equations without the v -constraint equation (see Section 2.5). When all functions depend on (ξ, η) with $\xi = \theta - \lambda v$ for some constant λ , we get a system of PDEs in two variables that is studied further in [1]. This system describes space-times that are stationary with respect to an observer moving close to the speed of light.

We also derive asymptotic equations for the diffraction of gravitational waves with general polarizations in two transverse directions. These equations are given by (3.12)–(3.14), (C.1), and (C.3)–(C.5), but they are much more complicated than the ones for plane-polarized waves written out above.

2 Geometrical optics

The Einstein equations do not form a hyperbolic system of PDEs because of their gauge-covariance, but they are hyperbolic when written with respect to a suitable gauge. In order to develop and interpret geometrical optics solutions for the Einstein equations, it is useful to begin by studying geometrical optics solutions for hyperbolic systems of variational wave equations.

We remark that there is an analogy between geometrical optics theories for hyperbolic systems of conservation laws [19] and variational wave equations. For example, the inviscid Burgers equation [12, 20] is the analog of the Hunter-Saxton equation (2.14), and the unsteady transonic small disturbance equation [18] is the analog of the two-dimensional Hunter-Saxton equation (2.15). One can also derive large-amplitude geometrical optics theories for linearly degenerate waves in hyperbolic conservation laws (see [33, 34], for example) that are analogous to the large-amplitude theories described here for the Einstein equations.

There are other nonlinear geometrical optics theories for dispersive waves, most notably Whitham's ‘averaged Lagrangian method’ [37, 38] for large-amplitude dispersive waves. These dispersive theories have a different character from the ones for nondispersive hyperbolic waves. Nonlinear dispersive waves have specific waveforms — given by traveling wave solutions — in which the effects of dispersion and nonlinearity balance, whereas nondispersive hyperbolic wave equations and the Einstein equations have traveling wave solutions with arbitrary waveforms, which may distort as the waves propagate.

In straightforward, non-diffractive theories of geometrical optics, waves are locally approximated by plane waves and propagate along rays. In linear geometrical optics, the wave amplitude satisfies an ODE (the transport equation) along a ray; in nonlinear geometrical optics, the amplitude-waveform function satisfies a nonlinear PDE in one space dimension (a generalization of the transport equation) along a ray. This difference arises because linear hyperbolic waves propagate without distortion, so one requires only an ODE along each ray to determine the change in the wave amplitude; by contrast, wave-steepening and other effects distort the waveform of nonlinear hyperbolic waves, so one requires a PDE along each ray to determine the change in the wave amplitude and the waveform.

Straightforward geometrical optics, and a local plane-wave approximation, break down when the effects of wave-diffraction become important; for example, this occurs when a high-frequency wave focuses at a caustic, or when a wave beam of large (relative to its wavelength), but finite, transverse extent spreads out. The effects of diffraction on a high-frequency wave may be described by the inclusion of additional length-scales in the straightforward geometrical optics asymptotic solution, and one then obtains asymptotic PDEs with a larger number of independent variables.

Perhaps the most basic asymptotic solution that incorporates the effect of wave diffraction is the ‘parabolic approximation’ described below for wave equations. In Section 3, we derive analogous asymptotic solutions of the Einstein equations that describe the diffraction of large-amplitude gravitational waves.

2.1 The wave equation

We begin by recalling geometrical optics theories for the linear wave equation

$$(2.1) \quad g_{tt} = \nabla \cdot (c_0^2 \nabla g).$$

Here, $g(t, x)$ is a scalar function, the wave speed $c_0(t, x)$ is a given smooth function, and $x \in \mathbb{R}^d$. Although well-known, these theories provide a useful background for our analysis of nonlinear variational wave equations and the Einstein equations.

We look for a short-wavelength asymptotic solution $g = g^\varepsilon$ of (2.1), depending on a small parameter ε , of the form

$$g^\varepsilon(t, x) \sim a\left(\frac{u(t, x)}{\varepsilon^2}, t, x\right) \quad \text{as } \varepsilon \rightarrow 0.$$

The solution depends on a ‘fast’ phase variable²

$$\theta = \frac{u}{\varepsilon^2}$$

and ‘slow’ space-time variables (t, x) .

²Here, we use u/ε^2 as a phase variable, rather than u/ε , for consistency with the diffractive expansion below.

One finds that the scalar-valued phase function $u(t, x)$ satisfies the eikonal equation

$$(2.2) \quad u_t^2 = c_0^2 |\nabla u|^2.$$

The amplitude-waveform function $a(\theta, t, x)$ satisfies the equation

$$(2.3) \quad a_{\theta v} + N a_\theta = 0,$$

where

$$(2.4) \quad \partial_v = u_t \partial_t - c_0^2 \nabla u \cdot \nabla$$

is a derivative along the rays associated with the phase u , and $N(t, x)$ is given by

$$(2.5) \quad N = \frac{1}{2} \{u_{tt} - \nabla \cdot (c_0^2 \nabla u)\}.$$

Equation (2.3) has solutions of the form

$$a(\theta, t, x) = A(t, x) F(\theta),$$

where $A(t, x)$ is a wave-amplitude that satisfies an ODE along a ray (the transport equation of linear geometrical optics)

$$(2.6) \quad A_v + N A = 0,$$

and $F(\theta)$ is an arbitrary function that describes the waveform of the wave. For example, if $F(\theta) = e^{i\theta}$, then the solution describes an oscillatory harmonic wave; if

$$F(\theta) = \begin{cases} \theta^n & \theta > 0, \\ 0 & \theta \leq 0, \end{cases}$$

then the solution describes a wavefront across which the normal derivative of g of order n jumps; if $F(\theta)$ has compact support, then the solution describes a pulse; and if $F(\theta)$ has different limits as $\theta \rightarrow \pm\infty$, then the solution describes a wave that carries a jump in g .

The term NA in the transport equation (2.6) describes the effect of the ray geometry on the wave amplitude. The coefficient N becomes infinite when rays focus at a caustic. The straightforward geometrical optics solution then becomes invalid, and diffractive effects must be taken into account [31].

There are many ways in which diffraction modifies straightforward geometrical optics. Here, we consider one of the simplest diffractive expansions, given by

$$g^\varepsilon(t, x) \sim a\left(\frac{u(t, x)}{\varepsilon^2}, \frac{y(t, x)}{\varepsilon}, t, x\right) \quad \text{as } \varepsilon \rightarrow 0.$$

This asymptotic solution depends upon an additional ‘intermediate’ variable

$$\eta = \frac{y}{\varepsilon},$$

where $y(t, x)$ is a scalar-valued transverse phase. This ansatz describes a high-frequency wave whose wavefronts are close to $u = \text{constant}$ that diffracts in the y -direction.

One finds that u satisfies the eikonal equation, as before, and $y_v = 0$, meaning that y is constant along the rays associated with u . Moreover, the amplitude-waveform function $a(\theta, \eta, t, x)$ satisfies the equation

$$(2.7) \quad a_{\theta v} + Na_{\theta} + \frac{1}{2}Da_{\eta\eta} = 0,$$

where the coefficient $D(t, x)$ of the diffractive term is given by

$$(2.8) \quad D = y_t^2 - c_0^2|\nabla y|^2.$$

For harmonic solutions, we have

$$a(\theta, \eta, t, x) = A(\eta, t, x)e^{i\theta},$$

and equation (2.7) reduces to a Schrödinger equation

$$i\{A_v + NA\} + \frac{1}{2}DA_{\eta\eta} = 0.$$

This ‘parabolic approximation’ and its generalizations are widely used in the study of wave propagation.

In the simplest case of the diffraction of plane wave solutions of the two-dimensional wave equation with wave speed $c_0 = 1$,

$$g_{tt} = g_{xx} + g_{yy},$$

we may choose

$$u = \frac{t-x}{\sqrt{2}}, \quad y = y, \quad v = \frac{t+x}{\sqrt{2}}.$$

Equation (2.7) is then

$$(2.9) \quad a_{\theta v} = \frac{1}{2}a_{\eta\eta}.$$

This equation describes waves that propagate in directions close to the positive x -direction, and is a wave equation in which θ and v are characteristic coordinates. We will see that equations with a similar structure to (2.9) in their highest-order derivatives arise from the Einstein equations.

2.2 A variational wave equation

Next, we consider the following nonlinear, scalar wave equation [21]

$$(2.10) \quad g_{tt} - \nabla \cdot (c^2(g)\nabla g) + c(g)c'(g)|\nabla g|^2 = 0,$$

where a prime denotes the derivative with respect to g and we assume that the wave speed $c: \mathbb{R} \rightarrow \mathbb{R}^+$ is a smooth, non-vanishing function. This equation is derived from the variational principle

$$\delta \int \left\{ \frac{1}{2}g_t^2 - \frac{1}{2}c^2(g)|\nabla g|^2 \right\} dt dx = 0.$$

The global existence of conservative weak solutions of (2.10) in one space-dimension is established in [8].

The structure of the nonlinear terms in (2.10) resembles that of the Einstein equations, although the effects of nonlinearity are qualitatively different because of the linear degeneracy of the Einstein equations.

We look for an asymptotic solution of (2.10) of the form [16, 19, 21]

$$g^\varepsilon(t, x) \sim g_0(t, x) + \varepsilon^2 a\left(\frac{u(t, x)}{\varepsilon^2}, t, x\right) \quad \text{as } \varepsilon \rightarrow 0.$$

This solution represents a small-amplitude, high-frequency perturbation of a slowly-varying field g_0 . The amplitude and the wavelength of the perturbation are chosen to be of the same order of magnitude because this leads to a balance between the effects of weak nonlinearity and the ray geometry.

First, we suppose that the amplitude-waveform function $a(\theta, t, x)$ is a periodic function of the phase variable θ . We assume, without loss of generality, that its mean with respect to θ is zero.

We find that the phase u satisfies the eikonal equation (2.2) with $c_0 = c(g_0)$. The mean-field g_0 satisfies the nonlinear wave equation

$$(2.11) \quad g_{0tt} - \nabla \cdot (c_0^2 \nabla g_0) + c_0 c_0' |\nabla g_0|^2 + |\nabla u|^2 c_0 c_0' \langle a_\theta^2 \rangle = 0,$$

where the angular brackets denote an average with respect to θ over a period, and $c_0' = c'(g_0)$. Equation (2.11) has the same form as the original wave equation (2.10), with an additional source term proportional to the mean energy-density of the wave-field.

The amplitude-waveform function $a(\theta, t, x)$ satisfies the periodic Hunter-Saxton equation,

$$(2.12) \quad \left\{ a_v + \left(\frac{1}{2} \Lambda a^2 \right)_\theta + N a \right\}_\theta = \frac{1}{2} \Lambda \{ a_\theta^2 - \langle a_\theta^2 \rangle \},$$

where ∂_v is the ray derivative defined in (2.4), N is given by (2.5), and

$$(2.13) \quad \Lambda = -|\nabla u|^2 c_0 c_0'.$$

The coefficient Λ is proportional to the derivative of the wave-speed with respect to the wave amplitude, and it provides crucial information about the effect of nonlinearity on the waves; for a given g_0 , we obtain a nonlinear PDE for the amplitude-waveform function only when $\Lambda \neq 0$.

We may ensure that the higher-order terms in this expansion for waves with periodic waveforms also depend periodically on θ . As a result, no secular terms arise in the expansion, and it is uniformly valid as $\varepsilon \rightarrow 0$ for $(t, x) = O(1)$ and $\theta = O(\varepsilon^{-2})$. If we consider waves with localized waveforms, such as pulses or jumps, then we cannot in general eliminate secular terms in θ , but we may obtain an asymptotic solution that is valid near³ the wave-front $u = 0$ in which we neglect

³Specifically, the expansion is valid when $\theta = O(1)$, or $u = O(\varepsilon^2)$, and $t = O(1)$.

mean-field effects. Thus, the background field $g_0(t, x)$ is a solution of the original wave equation, and $a(\theta, t, x)$ satisfies

$$(2.14) \quad \left\{ a_\nu + \left(\frac{1}{2} \Lambda a^2 \right)_\theta + Na \right\}_\theta = \frac{1}{2} \Lambda a_\theta^2.$$

For results concerning the global existence of weak solutions of (2.14), see [7] and references given there.

It is straightforward to include diffractive effects in the expansion for localized waves. The asymptotic solution has the form

$$g^\varepsilon(t, x) \sim g_0(t, x) + \varepsilon^2 a \left(\frac{u(t, x)}{\varepsilon^2}, \frac{y(t, x)}{\varepsilon}, t, x \right) \quad \text{as } \varepsilon \rightarrow 0.$$

Here, g_0 is a solution of the original wave equation, the phase u satisfies the eikonal equation at g_0 , the transverse function y is constant along the rays associated with u , and $a(\theta, \eta, t, x)$ satisfies a (1 + 2)-dimensional generalization of the Hunter-Saxton equation (2.14),

$$(2.15) \quad \left\{ a_\nu + \left(\frac{1}{2} \Lambda a^2 \right)_\theta + Na \right\}_\theta + \frac{1}{2} D a_{\eta\eta} = \frac{1}{2} \Lambda a_\theta^2,$$

where D is given by (2.8).

2.3 Hyperbolic systems of variational wave equations

In this section, we consider a class of hyperbolic systems of nonlinear wave equations that are derived from a variational principle whose action is a quadratic function of the derivatives of the field with coefficients depending on the field [21]

$$(2.16) \quad \delta \int A_{pq}^{\alpha\beta}(g) \frac{\partial g^p}{\partial x^\alpha} \frac{\partial g^q}{\partial x^\beta} dx = 0.$$

Here, $x = (x^0, \dots, x^d) \in \mathbb{R}^{d+1}$ are space-time variables,

$$g = (g^1, \dots, g^m) \quad \mathbb{R}^{d+1} \rightarrow \mathbb{R}^m$$

are dependent variables, $A_{pq}^{\alpha\beta} \quad \mathbb{R}^m \rightarrow \mathbb{R}$ are smooth coefficient functions, and we use the summation convention. We assume that

$$A_{pq}^{\alpha\beta} = A_{pq}^{\beta\alpha} = A_{qp}^{\alpha\beta}.$$

The Euler-Lagrange equations associated with (2.16) are

$$(2.17) \quad G_p [g] = 0,$$

where

$$(2.18) \quad G_p [g] = \frac{\partial}{\partial x^\alpha} \left\{ A_{pq}^{\alpha\beta} (g) \frac{\partial g^q}{\partial x^\beta} \right\} - \frac{1}{2} \frac{\partial A_{qr}^{\alpha\beta}}{\partial g^p} (g) \frac{\partial g^q}{\partial x^\alpha} \frac{\partial g^r}{\partial x^\beta}.$$

We assume that (2.17) forms a hyperbolic system of PDEs. The scalar wave equation considered in the previous section is the simplest representative of this class of

equations. For discussions of recent local existence results for quasilinear systems of wave equations and the Einstein equations, see [5, 25].

The weakly nonlinear geometrical optics solution of (2.17) has the form [16, 19, 21]

$$(2.19) \quad g^\varepsilon(x) \sim g_0(x) + \varepsilon^2 a\left(\frac{u(x)}{\varepsilon^2}, x\right) R(x) \quad \text{as } \varepsilon \rightarrow 0,$$

where $g_0: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^m$ is a background field, $u: \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ is a phase function, $a: \mathbb{R} \times \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ is an amplitude-waveform function, and $R: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^m$ is a vector field.

The phase u satisfies the eikonal equation

$$(2.20) \quad \det[u_{x^\alpha} u_{x^\beta} A_{pq}^{\alpha\beta}(g_0)] = 0.$$

We introduce the singular $m \times m$ matrix C with components

$$(2.21) \quad C_{pq} = u_{x^\alpha} u_{x^\beta} A_{pq}^{\alpha\beta}(g_0).$$

Then the vector R in (2.19), with components R^p , is a null-vector of C , so that $C_{pq} R^q = 0$. Here, and in (2.19), we assume that we are dealing with a simple characteristic, meaning that the null-space of C is one-dimensional.

We write the scalar-valued amplitude-waveform function $a(\theta, x)$ as a function of the ‘fast’ phase variable $\theta = u/\varepsilon^2$ and the slow variables x . If $a(\theta, x)$ is a periodic function of θ with zero mean, then the mean-field $g_0(x)$ satisfies the equation

$$(2.22) \quad G_p[g_0] = \frac{1}{2} H_p \langle a_\theta^2 \rangle,$$

where the angular brackets denote an average with respect to θ , and

$$(2.23) \quad H_p = u_{x^\alpha} u_{x^\beta} \frac{\partial A_{qr}^{\alpha\beta}}{\partial g^p}(g_0) R^q R^r.$$

Equation (2.22) has the same form as the original equation (2.17) for g with an additional source term proportional to $\langle a_\theta^2 \rangle$. The H_p are interaction coefficients that describe the effect of the wave on the mean-field.

The equation for a is (2.12) with

$$(2.24) \quad \begin{aligned} \partial_v &= 2u_{x^\beta} A_{pq}^{\alpha\beta}(g_0) R^p R^q \partial_{x^\alpha}, \\ \Lambda &= u_{x^\alpha} u_{x^\beta} \frac{\partial A_{qr}^{\alpha\beta}}{\partial g^p}(g_0) R^p R^q R^r, \\ N &= \frac{\partial}{\partial x^\alpha} \{u_{x^\beta} A_{pq}^{\alpha\beta}(g_0) R^p R^q\} - u_{x^\alpha} \frac{\partial A_{qr}^{\alpha\beta}}{\partial g^p}(g_0) \frac{\partial g_0^r}{\partial x^\beta} R^p R^q. \end{aligned}$$

For localized waveforms, the mean-field interactions may be neglected, and we can obtain diffractive versions of this expansion as before.

These equations generalize easily to the case when (2.17) has a multiple characteristic of constant multiplicity $n \geq 2$. In that case, one obtains a mean-field equation whose source term is a sum of averages of products of θ -derivatives of the

amplitude-waveform functions, and an $n \times n$ system of Hunter-Saxton equations for the amplitude-waveform functions. In particular, if the vectors $\{R_1, \dots, R_n\}$ form a basis of the null-space of the matrix C defined in (2.21), then the coefficients Λ_{ijk} of the nonlinear terms in the system of equations for the amplitude-waveform functions are given by

$$(2.25) \quad \Lambda_{ijk} = u_{x^\alpha} u_{x^\beta} \frac{\partial A_{qr}^{\alpha\beta}}{\partial g^p} (g_0) R_i^p R_j^q R_k^r, \quad 1 \leq i, j, k \leq n.$$

2.4 Linearly degenerate wave equations

The coefficient Λ in (2.24) may be interpreted as a derivative of a wave speed with respect to the wave amplitude, as is explained further in [3]. Motivated by the corresponding definition for hyperbolic conservation laws introduced by Lax [28], we make the following definition.

Definition 2.1. A simple characteristic of a hyperbolic system of variational equations (2.17)–(2.18) is *genuinely nonlinear* (respectively, *linearly degenerate*) if, for every $g_0 \in \mathbb{R}^m$ and every non-zero $du \in \mathbb{R}^{d+1}$ belonging to the solution branch of (2.20) associated with the characteristic, the quantity Λ defined in (2.24) is non-zero (respectively, zero). A multiple characteristic of constant multiplicity n is *linearly degenerate* if all coefficients Λ_{ijk} defined in (2.25) are zero. We say that a system is *genuinely nonlinear* (respectively, *linearly degenerate*) if all of its characteristics are genuinely nonlinear (respectively, linearly degenerate).

Thus, for linearly degenerate wave equations, the amplitude-waveform function in the weakly nonlinear theory corresponding to the ansatz (2.19) satisfies a linear PDE. Quadratically nonlinear interactions with a mean-field may still occur, however: it follows from (2.23)–(2.24) that $\Lambda = H_p R^p$, so we may have $\Lambda = 0$ but $H_p \neq 0$.

Even if a characteristic of a nonlinear wave equation is not linearly degenerate, a loss of genuine nonlinearity may occur in which Λ vanishes at a particular point $g_0 \in \mathbb{R}^m$ and direction $du \in \mathbb{R}^{d+1}$.

For example, from (2.13), the scalar wave equation (2.10) is genuinely nonlinear if $c'(g) \neq 0$ for all $g \in \mathbb{R}$, has a loss of genuine nonlinearity if $c'(g) = 0$ for some $g \in \mathbb{R}$, and is linearly degenerate if c is constant, when it reduces to the linear wave equation. Systems of wave equations may be linearly degenerate without reducing to linear PDEs.

We emphasize that these definitions of genuine nonlinearity and linear degeneracy for wave equations are analogous to but different from the definitions for hyperbolic conservation laws. If a genuinely nonlinear variational wave equation for g were rewritten as a first-order hyperbolic system for g and $F = \partial g$, it would be classified as a linearly degenerate first-order hyperbolic system since the wave speeds of the original wave equation are independent of the derivative ∂g . For further discussion of linearly degenerate hyperbolic systems see [6].

Our interest here in linearly degenerate variational systems of wave equations is that they may possess large-amplitude, non-distorting, plane wave solutions. In that case, weakly nonlinear geometrical optics does not capture the nonlinear self-interaction of a wave, and one requires a strongly nonlinear theory. Non-distorting plane wave solutions exist only if the equations satisfy an additional degeneracy condition, namely that the coefficients H_p in (2.23) vanish on the plane wave solutions. We study these questions for general variational systems in [3]. Here, we discuss the corresponding theory for the Einstein equations.

2.5 The Einstein equations

The Einstein equations do not fall exactly into the class of variational wave equations defined in Section 2.3 because of their gauge-covariance. They can be derived from a variational principle of the form (2.16), obtained after an integration by parts in the Einstein-Hilbert action, but the resulting Euler-Lagrange equations are not hyperbolic. Nevertheless — as is well-known — they become hyperbolic when written with respect to suitable coordinates, such as wave (or harmonic) coordinates. The equation for the metric \mathbf{g} , with components $g_{\mu\nu}$, then adopts a similar form to (2.17)–(2.18), namely

$$\frac{\partial}{\partial x^\alpha} \left\{ g^{\alpha\beta} \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right\} - H_{\mu\nu}(\mathbf{g})(\partial\mathbf{g}, \partial\mathbf{g}) = 0,$$

where $H_{\mu\nu}$ is a quadratic form in the metric derivatives $\partial\mathbf{g}$ with coefficients depending on \mathbf{g} . (See [30] for an explicit expression.)

The weakly nonlinear expansion described in Section 2.3 for variational systems of wave equations corresponds to the expansion of Choquet-Bruhat [11] and Isaacson [22] for the Einstein equations. The Einstein equations have multiple characteristics, so one obtains a system of equations for the amplitude-waveform functions, but all of the nonlinear coefficients (2.25) in these equations are zero. Thus, in this sense, the Einstein equations form a linearly degenerate system of variational wave equations.

Physically, the linear degeneracy is a consequence of the transverse nature of gravitational waves. The velocity of a gravitational wave that propagates in the x -direction is determined by the (t, x) -metric components, but (in a suitable transverse gauge) the wave carries variations in the (y, z) -metric components. Thus, the wave amplitude does not affect the wave velocity.

The linear degeneracy of the Einstein equations is related to the ‘polarized null condition’ of Choquet-Bruhat [13, 14] and the ‘weak null condition’ of Lindblad and Rodnianski [29]. This condition is what permits the existence of global smooth small-amplitude perturbations of Minkowski space-time [15, 24, 30], for example, and similar results would not be true for $(1 + 3)$ -dimensional variational wave equations in which $\Lambda \neq 0$. See also [4] for related global existence results for other quasilinear wave equations.

The Einstein equations possess non-distorting, plane wave solutions for large-amplitude gravitational waves, and these solutions form the basis of a strongly nonlinear geometrical optics theory for large-amplitude gravitational waves. It does not, however, appear possible to obtain a self-consistent theory for oscillatory large-amplitude waves, since the mean energy-momentum associated with an extended wave-packet would generate a very strong background curvature of space-time. This restriction is related to the fact that mean-field interactions with oscillatory waves already occur at leading order in the small-amplitude theory. We therefore consider localized waves.

An asymptotic theory for the propagation of localized, large-amplitude, rapidly varying gravitational waves into slowly varying space-times was developed in [2]. In this theory, the metric of a plane-polarized wave may be written with respect to a suitable coordinate system (u, v, y, z) as

$$(2.26) \quad \mathbf{g} = -2e^{-M} du dv + e^{-U+V} dy^2 + e^{-U-V} dz^2 + O(\varepsilon^2).$$

Here, the leading-order metric component functions U, V, M depend on θ, v , where the ‘fast’ phase variable θ is given by

$$\theta = \frac{u}{\varepsilon^2}.$$

The metric components may also depend on y, z , but these variables occur as parameters in the asymptotic equations.

Then (U, V, M) are functions of (θ, v) which satisfy the following PDEs:

$$(2.27) \quad U_{\theta\theta} - \frac{1}{2}(U_{\theta}^2 + V_{\theta}^2) + U_{\theta}M_{\theta} = 0,$$

$$(2.28) \quad U_{\theta v} - U_{\theta}U_v = 0,$$

$$(2.29) \quad V_{\theta v} - \frac{1}{2}(U_{\theta}V_v + U_vV_{\theta}) = 0,$$

$$(2.30) \quad M_{\theta v} + \frac{1}{2}(U_{\theta}U_v - V_{\theta}V_v) = 0.$$

Equations (2.28)–(2.30) are wave equations for (U, V, M) in characteristic coordinates (θ, v) , and (2.27) is a constraint which is preserved by (2.28)–(2.30).

These equations correspond to a well-known exact solution of the Einstein equations, the colliding plane wave solution [17, 23, 35, 36], without the usual constraint equation in v ,

$$U_{vv} - \frac{1}{2}(U_v^2 + V_v^2) + U_vM_v = 0.$$

This constraint need not be satisfied if the slowly-varying space-time into which the wave propagates is not that of a counter-propagating gravitational wave. If it is not satisfied, then the resulting metric is an asymptotic solution of the Einstein equations but not an exact solution.

The asymptotic equations we derive in this paper are a diffractive generalization of (2.27)–(2.30).

3 The asymptotic expansion

In this section, we outline our asymptotic expansion of the Einstein equations, and specialize it to the case of plane-polarized gravitational waves that diffract in a single direction. Our goal is to explain the structure of the expansion and the resulting perturbation equations. The detailed algebra is summarized in the appendices.

3.1 The general expansion

Let \mathbf{g} be a Lorentzian metric. We denote the covariant components of \mathbf{g} with respect to a local coordinate system x^α by $g_{\alpha\beta}$. The connection coefficients $\Gamma^\lambda_{\alpha\beta}$ and the covariant components $R_{\alpha\beta}$ of the Ricci curvature tensor associated with \mathbf{g} are defined by

$$(3.1) \quad \Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} \left(\frac{\partial g_{\beta\mu}}{\partial x^\alpha} + \frac{\partial g_{\alpha\mu}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \right),$$

$$(3.2) \quad R_{\alpha\beta} = \frac{\partial \Gamma^\lambda_{\alpha\beta}}{\partial x^\lambda} - \frac{\partial \Gamma^\lambda_{\beta\lambda}}{\partial x^\alpha} + \Gamma^\lambda_{\alpha\beta} \Gamma^\mu_{\lambda\mu} - \Gamma^\mu_{\alpha\lambda} \Gamma^\lambda_{\beta\mu}.$$

Here, and below, Greek indices $\alpha, \beta, \lambda, \mu$ run over the values 0, 1, 2, 3. The vacuum Einstein equations may be written as

$$(3.3) \quad R_{\alpha\beta} = 0.$$

We look for asymptotic solutions of (3.3) with metrics of the form

$$(3.4) \quad \mathbf{g} = \mathbf{g} \left(\frac{u(x^\alpha)}{\varepsilon^2}, \frac{y^a(x^\alpha)}{\varepsilon}, x^\alpha; \varepsilon \right),$$

$$\mathbf{g}(\theta, \eta^a, x^\alpha; \varepsilon) = \mathbf{g}^0(\theta, \eta^a, x^\alpha) + \varepsilon \mathbf{g}^1(\theta, \eta^a, x^\alpha) + \varepsilon^2 \mathbf{g}^2(\theta, \eta^a, x^\alpha) + O(\varepsilon^3).$$

Here, ε is a small parameter, u is a phase, y^a with $a = 2, 3$ are transverse variables, and

$$\theta = \frac{u}{\varepsilon^2}, \quad \eta^a = \frac{y^a}{\varepsilon}$$

are ‘stretched’ variables. The ansatz in (3.4) corresponds to a metric that varies rapidly and strongly in the u -direction, with less rapid variations in the y^a -directions, and slow variations in x^α .

We remark that the form of this ansatz is relative to a class of local coordinate systems, since an ε -dependent change of coordinates can alter the way in which the metric depends on ε . It would be desirable to give a geometrically intrinsic characterization of such an ansatz and to carry out the expansion in a global, coordinate-invariant way, but we do not attempt to do so here.

We use (3.4) in the Einstein equations (3.3), expanding derivatives of a function f with respect to x^μ as

$$(3.5) \quad \frac{\partial}{\partial x^\mu} f \left(\frac{u}{\varepsilon^2}, \frac{y^a}{\varepsilon}, x^\alpha \right) = \frac{1}{\varepsilon^2} f_{,\theta} u_\mu + \frac{1}{\varepsilon} f_{,\bar{a}} y_\mu^a + f_{,\mu},$$

and then treating (θ, η^a, x^a) as independent variables. In (3.5) and below, we use the shorthand notation

$$(3.6) \quad f_{,\theta} = \left. \frac{\partial f}{\partial \theta} \right|_{\eta^a, x^a}, \quad f_{,\bar{a}} = \left. \frac{\partial f}{\partial \eta^a} \right|_{\theta, x^a}, \quad f_{,\mu} = \left. \frac{\partial f}{\partial x^\mu} \right|_{\theta, \eta^a},$$

$$u_\mu = \frac{\partial u}{\partial x^\mu}, \quad y_\mu^a = \frac{\partial y^a}{\partial x^\mu}.$$

We will omit commas from derivatives when this does not lead to any confusion.

Using the expansions of the connection coefficients and the Ricci curvature components given in Appendix A, we find that to get a non-trivial asymptotic solution of (3.3) at the leading orders, the phase u must be null to leading order, and the transverse variables y^a must be constant along the rays associated with u to leading order. Moreover, the leading-order metric must have the form of the colliding plane wave metric.

To carry out the expansion in detail, we use the gauge-covariance of the Einstein equations to make a choice of coordinates that is adapted to the metric in (3.4). We assume that u is approximately null up to the order ε^2 and that y^2, y^3 are constant along the rays associated with u up to the order ε . Furthermore, we assume that we can extend (u, y^2, y^3) to a local coordinate system

$$(x^0, x^1, x^2, x^3) = (u, v, y^2, y^3).$$

Then, as shown in Appendix B, we can use appropriate gauge transformations, which involve a near identity transformation of the phase and the transverse variables, to write a general metric (3.4) as

$$(3.7) \quad \mathbf{g} = 2 \overset{0}{g}_{01} dx^0 dx^1 + \overset{0}{g}_{ab} dx^a dx^b$$

$$+ \varepsilon \left\{ 2 \overset{1}{g}_{1a} dx^1 dx^a + \overset{1}{g}_{ab} dx^a dx^b \right\} + \varepsilon^2 \overset{2}{g}_{ij} dx^i dx^j + O(\varepsilon^3).$$

In (3.7) and below, indices i, j, k, \dots take on the values 1, 2, 3, while indices a, b, c, \dots take on the values 2, 3. We raise and lower indices using the leading order metric components; for example, we write

$$\left(\overset{0}{g}{}^{a\beta} \right) = \left(\overset{0}{g}{}_{a\beta} \right)^{-1}, \quad h^a = \overset{0}{g}{}^{a\beta} h_\beta.$$

We also show in Appendix B that we have one additional gauge freedom in (3.7), involving a nonlinear transformation of the phase, which we can use to set either

$$\overset{0}{g}_{01} = 1$$

or one of the three components

$$\overset{1}{g}_{12}, \quad \overset{1}{g}_{13}, \quad \overset{2}{g}_{11}$$

equal to zero. We will exploit this gauge freedom later, since it is convenient in formulating a variational principle for the asymptotic equations (see Appendix D).

We use (3.7) in (3.1)–(3.2) and expand the result with respect to ε . We find that

$$(3.8) \quad R_{\alpha\beta} = \frac{1}{\varepsilon^4} R_{\alpha\beta}^{-4} + \frac{1}{\varepsilon^3} R_{\alpha\beta}^{-3} + \frac{1}{\varepsilon^2} R_{\alpha\beta}^{-2} + O(\varepsilon^{-1}),$$

where explicit expressions for the nonzero components of the Ricci tensor up to the order ε^{-2} are summarized in Appendix C.

Using (3.8) in (3.3) and equating coefficients of ε^{-4} , ε^{-3} and ε^{-2} to zero, we get

$$(3.9) \quad R_{\alpha\beta}^{-4} = 0,$$

$$(3.10) \quad R_{\alpha\beta}^{-3} = 0,$$

$$(3.11) \quad R_{\alpha\beta}^{-2} = 0.$$

These perturbation equations lead to a closed set of equations for the leading-order and first-order components of the metric, as we now explain.

The only component of (3.9) that is not identically satisfied is

$$(3.12) \quad R_{00}^{-4} = 0.$$

From (3.12) and (C.1), it follows that $g_{\alpha\beta}^0$ satisfies

$$-\frac{1}{2}(g^{ab} g_{ab,\theta})_{,\theta} + \frac{1}{2} g^{01} g_{01,\theta} g^{ab} g_{ab,\theta} - \frac{1}{4} g^{ac} g_{bc,\theta} g^{bd} g_{ad,\theta} = 0.$$

This equation is a constraint on the leading order metric which involves only θ -derivatives, and has the same form as the constraint equation for a single gravitational plane wave.

The nonzero components of the Ricci tensor at the next order in ε are

$$R_{00}^{-3}, \quad R_{0a}^{-3}.$$

The condition

$$(3.13) \quad R_{0a}^{-3} = 0,$$

yields equations for g_{1a}^1 and their derivatives with respect to θ , with nonhomogeneous terms depending on θ - and η -derivatives of the leading order metric. Thus, (3.13) provides two equations relating the first-order perturbation in the metric to the leading order metric.

The equation

$$R_{00}^{-3} = 0$$

is a single equation that is homogeneous in the first-order components g_{ab}^1 , as can be seen from (C.2). A nonzero solution of this equation corresponds physically to a free, small-amplitude gravitational wave of strength of the order ε propagating in the space-time of the large-amplitude gravitational wave. Retaining a nonzero

solution of this homogenous equation would not change our final equations for the large-amplitude wave, and for simplicity we assume that

$${}^1g_{ab} = 0.$$

This assumption is also consistent with the higher-order perturbation equations.

At the next order in ε , the nonzero components of the Ricci curvature are

$${}^{-2}R_{00}, \quad {}^{-2}R_{01}, \quad {}^{-2}R_{0a}, \quad {}^{-2}R_{ab}.$$

The perturbation equations for ${}^{-2}R_{00}$ and ${}^{-2}R_{0a}$ in (3.11) give non-homogenous equations for the second-order metric components

$${}^2g_{ab}, \quad {}^2g_{1a},$$

with source terms depending on the lower-order components of the metric. These equations are satisfied by a suitable choice of the second-order metric components, and they are decoupled from the equations for the leading-order and first-order metric components. We therefore do not consider them further here.

The remaining perturbation equations are

$$(3.14) \quad {}^{-2}R_{01} = 0, \quad {}^{-2}R_{ab} = 0.$$

By use of the gauge freedom mentioned at the beginning of this section, we may set

$${}^2g_{11} = 0.$$

In that case, (3.14) provides a set of four equations relating the leading-order and first-order metric components, and their derivatives.

We remark that, in general, one cannot eliminate secular terms from the asymptotic solution that are unbounded as the ‘fast’ phase variable θ tends to infinity. As a result, the validity of the asymptotic equations is restricted to a thin layer of thickness of the order ε^2 about the hypersurface $u = 0$, where $\theta = O(1)$. A global asymptotic solution can be obtained by matching the ‘inner’ solution inside this layer with appropriate ‘outer’ solutions, such as slowly varying space-times on either side of the wave. In this paper, however, we focus on the construction of asymptotic solutions for localized gravitational waves, and do not consider any matching problems.

Summarizing these results, we find that after a suitable gauge transformation, an asymptotic solution of the Einstein equations (3.3) of the form (3.4) may be written as

$$(3.15) \quad \mathbf{g} = 2 {}^0g_{01} dx^0 dx^1 + {}^0g_{ab} dx^a dx^b + 2\varepsilon {}^1g_{1a} dx^1 dx^a + O(\varepsilon^2),$$

where $u = x^0$, $y^a = x^a$ and the six metric components

$${}^0g_{01}, \quad {}^0g_{ab}, \quad {}^1g_{1a}$$

satisfy a system (3.12)–(3.14) of seven equations. Equation (3.12) is an ODE in θ , and, in the case of plane-polarized waves, we show that it is a gauge-type constraint that is preserved by the remaining equations (3.13)–(3.14).

The explicit form of the equations follows from the expressions in (C.1), (C.3)–(C.5) for the corresponding components in the expansion of the Ricci tensor that appear in (3.12)–(3.14). In general, these equations are very complicated, but they simplify considerably in special cases.

3.2 Plane-polarized gravitational waves

In this section, we specialize our asymptotic solution to the case of a plane-polarized gravitational wave that diffracts in a single direction, and write out the resulting equations explicitly.

We choose coordinates

$$(3.16) \quad (x^0, x^1, x^2, x^3) = (u, v, y, z)$$

in which the metric has the form (3.15). We suppose that the metric depends on the variables θ, η, v where the phase variable θ and the transverse variable η are defined by

$$\theta = \frac{u}{\varepsilon^2}, \quad \eta = \frac{y}{\varepsilon},$$

and is independent of the second transverse variable $\zeta = z/\varepsilon$. This means that the wave diffracts only in the y -direction. We could also allow the metric to depend on z , which would appear in the final equations as a parameter.

We consider a leading-order metric that has the form of the colliding plane wave metric for a plane-polarized wave, polarized in the (y, z) -directions. We have seen that we need only retain the higher order components

$$g_{1a}^1, \quad g_{11}^2.$$

From (C.3), one component of (3.13) is homogeneous in g_{13}^1 , since there is no dependence on ζ , and we may assume that

$$g_{13}^1 = 0.$$

We therefore take the special form of the metric in (3.7) given by

$$(3.17) \quad \mathbf{g} = -2e^{-M} dudv + e^{-U+V} dy^2 + e^{-U-V} dz^2 \\ + 2\varepsilon Y e^{-M} dydv + \varepsilon^2 T e^{-M} dv^2 + O(\varepsilon^2),$$

where the functions (U, V, M, Y, T) depend on (θ, η, v) . We recall that we have the gauge-freedom to set either M, Y , or T equal to zero. In writing the equations, we choose to set $T = 0$. We then get the metric (1.2).

The asymptotic metric (3.17) must satisfy equations (3.12)–(3.14). First, using (3.17) in (3.12) and simplifying the result, we obtain, after some algebra, the θ -constraint equation (1.6).

Next, using (3.17) and (C.3), we find that the only nontrivial component of (3.13) is

$$(3.18) \quad R_{02}^{-3} = 0.$$

After the introduction of D_η , ϕ , ψ defined in (1.3)–(1.5) and some algebra, we find that (3.18) may be written as (1.7).

Finally, using (3.17) and (C.4)–(C.5), we find that the only nontrivial components of (3.14) are

$$(3.19) \quad R_{01}^{-2} = 0, \quad R_{22}^{-2} = 0, \quad R_{33}^{-2} = 0.$$

After some algebra, we may write these equations as (1.8)–(1.10). We have also checked the results by use of MAPLE.

3.3 The variational principle

Equations (3.12)–(3.14) can be derived from a variational principle, which is obtained by expanding the variational principle for the Einstein equations. In order to formulate a variational principle, it is necessary to retain the component

$$T = g_{11}^2,$$

which was set equal to zero in our previous choice of coordinates. Variations with respect to this component yield the gauge-type constraint (3.12), and it may be set to zero after taking variations with respect to it.

The general form of the asymptotic variational principle is given in Appendix D. Here, we specialize it to the case of a plane-polarized gravitational wave considered in Section 3.2. Using the metric (3.17) in (D.3) we find, after some algebra, that the variational principle (D.4) for (1.6)–(1.10) is

$$\delta S^{(1)} = 0, \quad S^{(1)} = \int L^{(-2)} d\theta dv d\eta,$$

where the Lagrangian $L^{(-2)}$ may be written as

$$(3.20) \quad L^{(-2)} = e^{-U} \left\{ 2M_{\theta v} + 4U_{\theta v} - V_\theta V_v - 3U_\theta U_v \right. \\ \left. - T_{\theta\theta} + T_\theta(M_\theta + 2U_\theta) + T(M_{\theta\theta} + 2U_{\theta\theta} - \frac{3}{2}U_\theta^2 - \frac{1}{2}V_\theta^2) \right. \\ \left. + e^{-(U+V+M)} \left[-2D_\eta\phi - D_\eta\psi + \frac{3}{2}\phi^2 + 2\phi\psi + \psi^2 \right] \right\}.$$

Variations of $S^{(1)}$ with respect to the second-order metric component T lead to the θ -constraint equation (1.6). Variations with respect to the first-order metric component Y , which appears in ϕ , ψ , and D_η , lead to equation (1.7). Variations with respect to U , V , M lead to the evolution equations (1.8)–(1.10), after we set $T = 0$.

4 Properties of the equations

In this section, we study some properties of the equations for plane-polarized waves that are derived in the previous section. We write out the structure of the highest-order derivatives that appear in the equations, and use this to formulate a reasonable IBVP for them. We also show that the θ -constraint equation is preserved by the evolution in v , and that the linearized equations are consistent with the linearized equations for gravitational waves in the parabolic approximation.

4.1 Structure of equations and an IBVP

In this subsection, we consider the structure of equations (1.6)–(1.10) in more detail. The first equation, (1.6), is an ODE with respect to θ relating (U, V, M) . As we show in the next section, this equation is a gauge-type constraint which holds for all v if it holds for $v = 0$, say. We may therefore neglect this equation provided that the initial data at $v = 0$ is compatible with it.

The remaining equations (1.7)–(1.10) form a system of equations for U, V, M, Y . In order to exhibit their structure, we rewrite them in a way that shows explicitly how the highest, second-order, derivatives appear.

Using (1.3)–(1.5), we may rewrite equation (1.7) as

$$(4.1) \quad Y_{\theta\theta} - \{(V + M)_{\theta} Y\}_{\theta} + \left\{ \frac{1}{2} U_{\theta}^2 - U_{\theta} V_{\theta} - \frac{1}{2} V_{\theta}^2 \right\} Y \\ = (U + V + M)_{\theta\eta} + M_{\eta} U_{\theta} - (U + V)_{\eta} V_{\theta}.$$

If U, V, M are assumed known, then this equation is a linear ODE in θ for Y .

We may rewrite equations (1.8)–(1.10) as

$$(4.2) \quad (U + V + M)_{\theta v} - \frac{1}{2} (U + V)_{\theta} (U + V)_v \\ = \frac{1}{2} e^{-(U+V+M)} \left\{ -\frac{1}{2} \phi^2 - \phi\psi \right\},$$

$$(4.3) \quad (U + V)_{\theta v} - \frac{1}{2} U_{\theta} (U + V)_v - \frac{1}{2} U_v (U + V)_{\theta}, \\ = \frac{1}{2} e^{-(U+V+M)} \{ D_{\eta}^2 (U + V) - \phi\psi - \psi^2 \}$$

$$(4.4) \quad U_{\theta v} - U_{\theta} U_v \\ = \frac{1}{2} e^{-(U+V+M)} \left\{ D_{\eta}^2 (U + V + M) - D_{\eta} (e^U Y_{\theta}) \right. \\ \left. - \frac{1}{2} \phi^2 - \phi\psi - \psi^2 \right\}.$$

Examining the terms that involve second order derivatives, we see that these equations consist of a $(1 + 2)$ -dimensional wave equation in (θ, η, v) for $(U + V)$, and two $(1 + 1)$ -dimensional wave equations in (θ, v) , for $(U + V + M)$ and U ,

in which (θ, v) are characteristic coordinates. An additional second-order derivative term, proportional to $D_\eta Y_\theta$, appears in the equation for U . The function Y is also coupled with the evolution equations through the dependence of the transverse derivative D_η , given in (1.3), on Y .

In order to specify a unique solution of these equations, we expect that we need to supplement the ODE (4.1) with data for Y and Y_θ on $\theta = 0$, say, and the evolution equations (4.2)–(4.4) with characteristic initial data for (U, V, M) on $\theta = 0$ and $v = 0$. Thus, a reasonable IBVP for (4.1)–(4.4) in the region $\theta > 0$, $-\infty < \eta < \infty$, and $v > 0$ is

$$\begin{aligned} (U, V, M) &= (U_0, V_0, M_0) && \text{on } v = 0, \\ (U, V, M) &= (U_1, V_1, M_1) && \text{on } \theta = 0, \\ (Y, Y_\theta) &= (Y_0, Y_1) && \text{on } \theta = 0. \end{aligned}$$

Here, (U_0, V_0, M_0) are given functions of (θ, η) that satisfy the constraint

$$U_{0\theta\theta} - \frac{1}{2} (U_{0\theta}^2 + V_{0\theta}^2) + U_{0\theta} M_{0\theta} = 0,$$

and $(U_1, V_1, M_1, Y_0, Y_1)$ are given compatible functions of (η, v) . This data may be interpreted as initial data for the state of the wave on the hypersurface $v = 0$, and boundary data on the leading wavefront $\theta = 0$.

4.2 The constraint equations

In this section, we show that the constraint equation (1.6) is preserved.

Proposition 4.1. *Suppose that (U, V, M, Y) are smooth functions that satisfy (1.7)–(1.10). Let*

$$(4.5) \quad F = U_{\theta\theta} - \frac{1}{2} (U_\theta^2 + V_\theta^2) + U_\theta M_\theta.$$

Then

$$(4.6) \quad F_v = U_v F.$$

Proof. We write the evolution equations (1.8)–(1.10) as

$$(4.7) \quad U_{\theta v} - U_\theta U_v = \frac{1}{2} e^{-(U+V+M)} A,$$

$$(4.8) \quad V_{\theta v} - \frac{1}{2} (U_\theta V_v + U_v V_\theta) = \frac{1}{2} e^{-(U+V+M)} B,$$

$$(4.9) \quad M_{\theta v} + \frac{1}{2} (U_\theta U_v - V_v V_\theta) = \frac{1}{2} e^{-(U+V+M)} C,$$

where, using the notation defined in (1.3)–(1.5),

$$(4.10) \quad A = D_\eta \phi + D_\eta \psi - \frac{1}{2} \phi^2 - \phi \psi - \psi^2,$$

$$(4.11) \quad B = -D_\eta \phi + \frac{1}{2} \phi^2,$$

$$(4.12) \quad C = -D_\eta \psi - \frac{1}{2} \phi^2 + \psi^2.$$

Differentiating (4.5) with respect to v , and using (4.7)–(4.9) and (4.5) to replace $U_{\theta v}$, $V_{\theta v}$, $M_{\theta v}$ and $U_{\theta\theta}$ in the result, we find that

$$(4.13) \quad F_v = U_v F + \frac{1}{2} e^{-(U+V+M)} D,$$

where

$$(4.14) \quad D = A_\theta - (U + V)_\theta A - V_\theta B + U_\theta C.$$

Differentiating equation (4.10) for A with respect to θ , introducing the commutator $[\partial_\theta, D_\eta]$ of ∂_θ and D_η , and using equations (1.5) and (1.7), we compute that

$$(4.15) \quad \begin{aligned} A_\theta &= \partial_\theta \left\{ D_\eta(\phi + \psi) - \frac{1}{2} \phi^2 - \phi \psi - \psi^2 \right\} \\ &= D_\eta(\phi + \psi)_\theta + [\partial_\theta, D_\eta](\phi + \psi) - (\phi + \psi)\phi_\theta - (\phi + 2\psi)\psi_\theta \\ &= D_\eta\{\psi(U + V)_\theta\} \\ &\quad - (\phi + \psi)\phi_\theta - (\phi + 2\psi)\psi_\theta + [\partial_\theta, D_\eta](\phi + \psi) \\ &= (U + V)_\theta D_\eta \psi + \psi D_\eta(U + V)_\theta \\ &\quad - (\phi + \psi)\phi_\theta - (\phi + 2\psi)\psi_\theta + [\partial_\theta, D_\eta](\phi + \psi) \\ &= (U + V)_\theta D_\eta \psi + \psi \partial_\theta D_\eta(U + V) - \psi [\partial_\theta, D_\eta](U + V) \\ &\quad - (\phi + \psi)\phi_\theta - (\phi + 2\psi)\psi_\theta + [\partial_\theta, D_\eta](\phi + \psi) \\ &= (U + V)_\theta D_\eta \psi + \psi \psi_\theta - (\phi + \psi)\phi_\theta - (\phi + 2\psi)\psi_\theta \\ &\quad + [\partial_\theta, D_\eta](\phi + \psi) - \psi [\partial_\theta, D_\eta](U + V) \\ &= (U + V)_\theta D_\eta \psi - (\phi + \psi)(\phi + \psi)_\theta \\ &\quad + [\partial_\theta, D_\eta](\phi + \psi) - \psi [\partial_\theta, D_\eta](U + V) \\ &= (U + V)_\theta \{D_\eta \psi - \psi(\phi + \psi)\} \\ &\quad + [\partial_\theta, D_\eta](\phi + \psi) - \psi [\partial_\theta, D_\eta](U + V). \end{aligned}$$

From the definition of D_η in (1.3), it follows that

$$[\partial_\theta, D_\eta] = U_\theta D_\eta + e^U Y_\theta \partial_\theta.$$

Hence, using (1.5) and (1.7), we get

$$\begin{aligned} [\partial_\theta, D_\eta](\phi + \psi) &= U_\theta D_\eta(\phi + \psi) + e^U Y_\theta(\phi + \psi)_\theta \\ &= U_\theta D_\eta(\phi + \psi) + \psi e^U Y_\theta(U + V)_\theta, \\ [\partial_\theta, D_\eta](U + V) &= U_\theta D_\eta(U + V) + e^U Y_\theta(U + V)_\theta \\ &= U_\theta \psi + e^U Y_\theta(U + V)_\theta. \end{aligned}$$

Using these equations in (4.15), and simplifying the result, we find that

$$A_\theta = (U + V)_\theta (D_\eta \psi - \phi \psi - \psi^2) + U_\theta (D_\eta \phi + D_\eta \psi - \psi^2).$$

Finally, using this equation and (4.10)–(4.12) in (4.14), and simplifying the result, we find that $D = 0$. It follows from (4.13) that F satisfies (4.6). \square

4.3 Linearization

We consider the small-amplitude limit of (1.6)–(1.10) in which

$$U, V, M, Y \rightarrow 0.$$

From the constraint equation (1.6), we have $U = O(V^2)$, so U is of higher-order in a linearized approximation and can be neglected, while V describes the wave-profile. (See §35.9–§35.10 in [32] for further discussion.)

From (1.3)–(1.4), we also have in this approximation that

$$\begin{aligned} D_\eta &= \partial_\eta, \\ \phi &= M_\eta - Y_\theta, \\ \psi &= V_\eta. \end{aligned}$$

Linearization of (1.7) yields

$$(\phi + \psi)_\theta = 0,$$

while linearization the evolution equations (1.8)–(1.10) gives

$$\begin{aligned} 0 &= \frac{1}{2}(\phi + \psi)_\eta, \\ V_{\theta v} &= -\frac{1}{2}\phi_\eta, \\ M_{\theta v} &= -\frac{1}{2}\psi_\eta. \end{aligned}$$

Neglecting functions of integration for simplicity, we find that these equations are satisfied if $M = -V$ and $Y = 0$. In that case, V satisfies an equation of the form (2.9):

$$V_{\theta v} = \frac{1}{2}V_{\eta\eta}.$$

The corresponding linearized metric is given by

$$\mathbf{g} = 2(1 - V)dudv + (1 + V)dy^2 + (1 - V)dz^2.$$

Linearization of the Einstein equations, with a suitable choice of gauge, leads to a set of linear wave equations for the metric components. One can verify that this linearization of our asymptotic solution agrees with what is obtained by an application of the parabolic approximation described in Section 2.1 to the linearized Einstein equations for plane-polarized waves.

Appendix A: Expansion of connection coefficients and Ricci tensor

In this appendix, we write out the expansions as $\varepsilon \rightarrow 0$ of the connection coefficients and the Ricci curvature components associated with a metric whose components have the form

$$g_{\alpha\beta} = g_{\alpha\beta} \left(\frac{u(x)}{\varepsilon^2}, \frac{y^a(x)}{\varepsilon}, x; \varepsilon \right),$$

$$g_{\alpha\beta}(\theta, \eta^a, x; \varepsilon) = g_{\alpha\beta}^0(\theta, \eta^a, x) + \varepsilon g_{\alpha\beta}^1(\theta, \eta^a, x) + \varepsilon^2 g_{\alpha\beta}^2(\theta, \eta^a, x) + O(\varepsilon^3).$$

The contravariant metric components $g^{\alpha\beta}$ satisfy

$$g^{\alpha\mu} g_{\mu\beta} = \delta_{\beta}^{\alpha}.$$

Expanding this equation in a power series in ε and solving for $g^{\alpha\beta}$, we get

$$(A.1) \quad g^{\alpha\beta} = g^{\alpha\beta}_0 - \varepsilon g^{\alpha\beta}_1 - \varepsilon^2 \left(g^{\alpha\beta}_2 - g^{\alpha\mu}_1 g^{\mu\nu}_0 g^{\alpha\nu}_1 \right) + O(\varepsilon^3),$$

$$= h^{\alpha\beta}_0 + \varepsilon h^{\alpha\beta}_1 + \varepsilon^2 h^{\alpha\beta}_2 + O(\varepsilon^3),$$

where $g^{\alpha\beta}_0$ is the inverse of $g_{\alpha\beta}^0$. Here and below, we use the leading order metric components to raise indices, so that

$$g^{\alpha\beta}_1 = g^{\alpha\mu}_0 g^{\beta\nu}_0 g^{\mu\nu}_1, \quad g^{\alpha\beta}_2 = g^{\alpha\mu}_0 g^{\beta\nu}_0 g^{\mu\nu}_2.$$

We note that, with this convention, the first-order term in the expansion of the contravariant metric components $g^{\alpha\beta}$ with respect to ε is

$$h^{\alpha\beta}_1 = -g^{\alpha\beta}_0,$$

not $g^{\alpha\beta}_1$.

We use the notation for derivatives in (3.6). The expansion of the connection coefficients (or Christoffel symbols) given in (3.1) is

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{\varepsilon^2} \Gamma^{-2\lambda}_{\alpha\beta} + \frac{1}{\varepsilon} \Gamma^{-1\lambda}_{\alpha\beta} + \Gamma^0_{\alpha\beta} + O(\varepsilon),$$

where

$$\begin{aligned}
\bar{\Gamma}^{-2\lambda}_{\alpha\beta} &= \frac{1}{2} \bar{g}^{\lambda\mu} \left(\bar{g}_{\beta\mu,\theta} u_\alpha + \bar{g}_{\alpha\mu,\theta} u_\beta - \bar{g}_{\alpha\beta,\theta} u_\mu \right), \\
\bar{\Gamma}^{-1\lambda}_{\alpha\beta} &= \frac{1}{2} \bar{g}^{\lambda\mu} \left(\bar{g}_{\beta\mu,\bar{a}} y_\alpha^a + \bar{g}_{\alpha\mu,\bar{a}} y_\beta^a - \bar{g}_{\alpha\beta,\bar{a}} y_\mu^a \right) \\
&\quad + \frac{1}{2} \bar{g}^{\lambda\mu} \left(\bar{g}_{\beta\mu,\theta} u_\alpha + \bar{g}_{\alpha\mu,\theta} u_\beta - \bar{g}_{\alpha\beta,\theta} u_\mu \right) \\
&\quad - \frac{1}{2} \bar{g}^{\lambda\mu} \left(\bar{g}_{\beta\mu,\theta} u_\alpha + \bar{g}_{\alpha\mu,\theta} u_\beta - \bar{g}_{\alpha\beta,\theta} u_\mu \right), \\
\bar{\Gamma}^0_{\alpha\beta} &= \frac{1}{2} \bar{g}^{\lambda\mu} \left(\bar{g}_{\beta\mu,\alpha} + \bar{g}_{\alpha\mu,\beta} - \bar{g}_{\alpha\beta,\mu} \right) \\
&\quad + \frac{1}{2} \bar{g}^{\lambda\mu} \left(\bar{g}_{\beta\mu,\bar{a}} y_\alpha^a + \bar{g}_{\alpha\mu,\bar{a}} y_\beta^a - \bar{g}_{\alpha\beta,\bar{a}} y_\mu^a \right) \\
&\quad - \frac{1}{2} \bar{g}^{\lambda\mu} \left(\bar{g}_{\beta\mu,\bar{a}} y_\alpha^a + \bar{g}_{\alpha\mu,\bar{a}} y_\beta^a - \bar{g}_{\alpha\beta,\bar{a}} y_\mu^a \right) \\
&\quad + \frac{1}{2} \bar{g}^{\lambda\mu} \left(\bar{g}_{\beta\mu,\theta} u_\alpha + \bar{g}_{\alpha\mu,\theta} u_\beta - \bar{g}_{\alpha\beta,\theta} u_\mu \right) \\
&\quad - \frac{1}{2} \bar{g}^{\lambda\mu} \left(\bar{g}_{\beta\mu,\theta} u_\alpha + \bar{g}_{\alpha\mu,\theta} u_\beta - \bar{g}_{\alpha\beta,\theta} u_\mu \right) \\
&\quad + \frac{1}{2} \bar{h}^{\lambda\mu} \left(\bar{g}_{\beta\mu,\theta} u_\alpha + \bar{g}_{\alpha\mu,\theta} u_\beta - \bar{g}_{\alpha\beta,\theta} u_\mu \right).
\end{aligned}$$

The expansion of the Ricci tensor is

$$R_{\alpha\beta} = \frac{1}{\varepsilon^4} \bar{R}^{-4}_{\alpha\beta} + \frac{1}{\varepsilon^3} \bar{R}^{-3}_{\alpha\beta} + \frac{1}{\varepsilon^2} \bar{R}^{-2}_{\alpha\beta} + O(\varepsilon^{-1}),$$

where

$$\begin{aligned}
\bar{R}^{-4}_{\alpha\beta} &= \bar{\Gamma}^{-2\mu}_{\alpha\beta,\theta} u_\mu - \bar{\Gamma}^{-2\mu}_{\beta\mu,\theta} u_\alpha + \bar{\Gamma}^{-2\mu}_{\alpha\beta} \bar{\Gamma}^{-2\nu}_{\mu\nu} - \bar{\Gamma}^{-2\mu}_{\alpha\nu} \bar{\Gamma}^{-2\nu}_{\beta\mu}, \\
\bar{R}^{-3}_{\alpha\beta} &= \bar{\Gamma}^{-1\mu}_{\alpha\beta,\theta} u_\mu - \bar{\Gamma}^{-1\mu}_{\beta\mu,\theta} u_\alpha + \bar{\Gamma}^{-2\mu}_{\alpha\beta,\bar{a}} y_\mu^a - \bar{\Gamma}^{-2\mu}_{\beta\mu,\bar{a}} y_\alpha^a \\
&\quad + \bar{\Gamma}^{-2\mu}_{\alpha\beta} \bar{\Gamma}^{-1\nu}_{\mu\nu} + \bar{\Gamma}^{-1\mu}_{\alpha\beta} \bar{\Gamma}^{-2\nu}_{\mu\nu} - \bar{\Gamma}^{-2\mu}_{\alpha\nu} \bar{\Gamma}^{-1\nu}_{\beta\mu} - \bar{\Gamma}^{-1\mu}_{\alpha\nu} \bar{\Gamma}^{-2\nu}_{\beta\mu}, \\
\bar{R}^{-2}_{\alpha\beta} &= \bar{\Gamma}^0_{\alpha\beta,\theta} u_\mu - \bar{\Gamma}^0_{\beta\mu,\theta} u_\alpha + \bar{\Gamma}^{-1\mu}_{\alpha\beta,\bar{a}} y_\mu^a - \bar{\Gamma}^{-1\mu}_{\beta\mu,\bar{a}} y_\alpha^a \\
&\quad + \bar{\Gamma}^{-2\mu}_{\alpha\beta,\mu} - \bar{\Gamma}^{-2\mu}_{\beta\mu,\alpha} + \bar{\Gamma}^{-2\mu}_{\alpha\beta} \bar{\Gamma}^0_{\mu\nu} + \bar{\Gamma}^0_{\alpha\beta} \bar{\Gamma}^{-2\nu}_{\mu\nu} \\
&\quad - \bar{\Gamma}^{-2\mu}_{\alpha\nu} \bar{\Gamma}^0_{\beta\mu} - \bar{\Gamma}^0_{\alpha\nu} \bar{\Gamma}^{-2\nu}_{\beta\mu} + \bar{\Gamma}^{-1\mu}_{\alpha\beta} \bar{\Gamma}^{-1\nu}_{\mu\nu} - \bar{\Gamma}^{-1\mu}_{\alpha\nu} \bar{\Gamma}^{-1\nu}_{\beta\mu}.
\end{aligned}$$

Appendix B: Gauge transformations

We consider a Lorentzian metric \mathbf{g} and local coordinates x^α in which

$$\mathbf{g} = g_{\alpha\beta} dx^\alpha dx^\beta.$$

We denote the contravariant form of \mathbf{g} by

$$\mathbf{g}^\sharp = g^{\alpha\beta} \partial_\alpha \partial_\beta.$$

We suppose that the metric \mathbf{g} depends upon a small parameter ε , and has the asymptotic expansion (3.4) as $\varepsilon \rightarrow 0$ where u , y^a , $a = 2, 3$, are independent functions. Furthermore, we assume that

$$(B.1) \quad \mathbf{g}^\sharp(du, du) = O(\varepsilon^2),$$

$$(B.2) \quad \mathbf{g}^\sharp(du, dy^a) = O(\varepsilon), \quad a = 2, 3.$$

The first condition states that du is an approximate null form up to the order ε^2 . The second condition states that y^a is constant along the rays associated with u up to the order ε .

In this appendix, we show that under these assumptions, after a near-identity transformation of the phase u and the transverse variables y^a , we can choose local coordinates x^α in which $u = x^0$, $y^a = x^a$ and the metric adopts the form in (3.7):

$$(B.3) \quad \mathbf{g} = 2 \overset{0}{g}_{01} dx^0 dx^1 + \overset{0}{g}_{ab} dx^a dx^b \\ + \varepsilon \left\{ 2 \overset{1}{g}_{1a} dx^1 dx^a + \overset{1}{g}_{ab} dx^a dx^b \right\} + \varepsilon^2 \overset{2}{g}_{ij} dx^i dx^j + O(\varepsilon^3).$$

As before, indices α, β, \dots take on the values 0, 1, 2, 3; indices i, j, k, \dots take on the values 1, 2, 3; and indices a, b, c, \dots take on the values 2, 3.

We also show that by a nonlinear transformation of the phase u , we can impose any one of the following four conditions:

$$\overset{0}{g}_{01} = 1, \quad \overset{1}{g}_{12} = 0, \quad \overset{1}{g}_{13} = 0, \quad \overset{2}{g}_{11} = 0.$$

To begin with, given a metric of the form (3.4), we choose a local coordinate system x^α in which

$$(B.4) \quad x^0 = u, \quad x^a = y^a.$$

Then (B.1)–(B.2) are equivalent to

$$(B.5) \quad \overset{0}{g}{}^{00} = \overset{1}{g}{}^{00} = 0, \quad \overset{0}{g}{}^{0a} = 0.$$

These conditions imply that

$$(B.6) \quad \overset{0}{g}_{11} = 0, \quad \overset{0}{g}_{1a} = 0.$$

Moreover, we must have

$$(B.7) \quad \det \overset{0}{g}_{ab} > 0,$$

which implies that the y^a are space-like coordinates.

The expansion (3.4) and the conditions (B.4), (B.6) are invariant under coordinate transformations of the form

$$x^\alpha \rightarrow x^\alpha + \varepsilon^2 \Psi^\alpha \left(\frac{x^0}{\varepsilon^2}, \frac{x^a}{\varepsilon}, x; \varepsilon \right),$$

$$\Psi^0(\theta, \eta^a, x; \varepsilon) = \varepsilon \Psi^0(\theta, \eta^a, x) + \varepsilon^2 \Psi^0(\theta, \eta^a, x) + O(\varepsilon^3),$$

$$\Psi^i(\theta, \eta^a, x; \varepsilon) = \Psi^i(\theta, \eta^a, x) + \varepsilon \Psi^i(\theta, \eta^a, x) + \varepsilon^2 \Psi^i(\theta, \eta^a, x) + O(\varepsilon^3).$$

Here, and below, we write

$$\theta = \frac{x^0}{\varepsilon^2}, \quad \eta^a = \frac{x^a}{\varepsilon}.$$

First, we make a change of coordinates of the form

$$x^0 \rightarrow x^0, \quad x^i \rightarrow x^i + \varepsilon^2 \Psi^i(\theta, \eta^a, x).$$

At the order zero in ε , the metric components transform according to

$$\begin{aligned} g_{00}^0 &\rightarrow g_{00}^0 + 2 \Psi_{,\theta}^i g_{0i}^0 + \Psi_{,\theta}^i \Psi_{,\theta}^j g_{ij}^0, \\ g_{0i}^0 &\rightarrow g_{0i}^0 + \Psi_{,\theta}^j g_{ij}^0, \\ g_{ij}^0 &\rightarrow g_{ij}^0, \end{aligned}$$

where we use the notation in (3.6) for θ -derivatives. From (B.7), the matrix g_{ab}^0 is invertible, and therefore we can choose $\Psi_{,\theta}^i$ so that

$$g_{00}^0 = 0, \quad g_{0a}^0 = 0.$$

In that case, the zero-order metric \mathbf{g}^0 takes the form given in (B.3).

Second, we make a change of coordinates of the form

$$x^\alpha \rightarrow x^\alpha + \varepsilon^3 \Psi^\alpha(\theta, \eta^a, x).$$

The metric (B.3) is unchanged at the order zero under the action of this transformation. At the order one, the components transform according to

$$\begin{aligned} g_{00}^1 &\rightarrow g_{00}^1 + 2 \Psi_{,\theta}^1 g_{01}^0, \\ g_{01}^1 &\rightarrow g_{01}^1 + \Psi_{,\theta}^0 g_{01}^0, \\ g_{0a}^1 &\rightarrow g_{0a}^1 + \Psi_{,\theta}^b g_{ab}^0, \\ g_{ij}^1 &\rightarrow g_{ij}^1. \end{aligned}$$

Choosing $\overset{1}{\Psi}^a$ appropriately, we can use these transformations to make

$$\overset{1}{g}_{0\alpha} = 0.$$

Moreover, the condition (B.5) implies that

$$\begin{aligned} \overset{1}{g}_{11} &= - \left(\overset{0}{g}_{01} \right)^2 \overset{1}{g}^{00} \\ &= 0. \end{aligned}$$

In that case, the first-order metric $\overset{1}{\mathbf{g}}$ takes the form given in (B.3).

Third, we make a change of coordinates of the form

$$x^a \rightarrow x^a + \varepsilon^4 \overset{2}{\Psi}^a(\theta, \eta^a, x).$$

The metric (B.3) is unchanged at the orders zero and one. At the order two, the metric components transform according to

$$\begin{aligned} \overset{2}{g}_{00} &\rightarrow \overset{2}{g}_{00} + 2 \overset{2}{\Psi}_{,\theta}^1 \overset{0}{g}_{01}, \\ \overset{2}{g}_{01} &\rightarrow \overset{2}{g}_{01} + \overset{2}{\Psi}_{,\theta}^0 \overset{0}{g}_{01}, \\ \overset{2}{g}_{0a} &\rightarrow \overset{2}{g}_{0a} + \overset{2}{\Psi}_{,\theta}^b \overset{0}{g}_{ab}, \\ \overset{2}{g}_{ij} &\rightarrow \overset{2}{g}_{ij}. \end{aligned}$$

Choosing $\overset{2}{\Psi}^a$ appropriately, we can use these transformations to make

$$\overset{2}{g}_{0\alpha} = 0,$$

and the resulting metric then has the form given in (B.3).

We remark that the choice of $u = x^0$ for the phase and $y^a = x^a$ for the transverse variables means that the coordinate changes above imply a change of the phase u and the transverse variables y^a appearing in (3.4) of the form

$$u \rightarrow u + \varepsilon^3 \Phi \left(\frac{u}{\varepsilon^2}, \frac{y^a}{\varepsilon}, x; \varepsilon \right), \quad y^a \rightarrow y^a + \varepsilon^2 \Psi^a \left(\frac{u}{\varepsilon^2}, \frac{y^a}{\varepsilon}, x; \varepsilon \right).$$

These transformations correspond to a near-identity transformation of the ‘fast’ phase θ and ‘intermediate’ transverse variables η^a ,

$$(B.8) \quad \theta \rightarrow \theta + \varepsilon \Phi(\theta, \eta^a, x; \varepsilon), \quad \eta^a \rightarrow \eta^a + \varepsilon \Psi^a(\theta, \eta^a, x; \varepsilon).$$

This transformation of the phases leads to a gauge-equivalent metric, and does not result in any loss of generality in the asymptotic solution.

Next, we show that there is an additional gauge freedom in (B.3) that is related to a nonlinear transformation of the phase. We consider a phase transformation

$$u \rightarrow \varepsilon^2 \Phi \left(\frac{u}{\varepsilon^2}, \frac{y^a}{\varepsilon}, x \right),$$

with a corresponding transformation of the ‘fast’ phase

$$(B.9) \quad \theta \rightarrow \Phi(\theta, \eta^a, x).$$

Under this transformation, the metric in (3.4) transforms as

$$\mathbf{g}\left(\frac{u}{\varepsilon^2}, \frac{y^a}{\varepsilon}, x; \varepsilon\right) \rightarrow \mathbf{g}\left(\Phi\left(\frac{u}{\varepsilon^2}, \frac{y^a}{\varepsilon}, x\right), \frac{y^a}{\varepsilon}, x; \varepsilon\right).$$

The transformed metric has the same form as the original one with respect to the transformed phase. Consequently, we have the freedom within the asymptotic theory to make such a change of phase. The significant difference from the previous transformations is that (B.9) involves a leading-order, nonlinear transformation of the ‘fast’ phase, whereas (B.8) involves a near-identity transformation.

In view of (B.4), we therefore consider a change of coordinates of the form

$$(B.10) \quad x^0 \rightarrow \varepsilon^2 \Phi\left(\frac{x^0}{\varepsilon^2}, \frac{x^a}{\varepsilon}, x\right).$$

Up to the order ε^2 , this coordinate change acts only on the following components of the metric (B.3):

$${}^0g_{01} \rightarrow \Phi_{,\theta} {}^0g_{01}, \quad {}^1g_{1a} \rightarrow {}^1g_{1a} + \Phi_{,\bar{a}} {}^0g_{01}, \quad {}^2g_{11} \rightarrow {}^2g_{11} + 2\Phi_{,\theta} {}^0g_{01}.$$

Thus, we can use (B.10) to set ${}^0g_{01} = 1$ or one of the three components

$${}^1g_{12}, \quad {}^1g_{13}, \quad {}^2g_{11}$$

equal to zero. More generally, we can choose Φ so that any appropriate relationship involving these four components is satisfied. The choice

$${}^2g_{11} = 0$$

is convenient because then θ is a characteristic variable of the asymptotic equations, and they adopt a somewhat simpler form.

Finally, we record here the explicit contravariant form of the metric tensor corresponding to (B.3). We have

$$(B.11) \quad \mathbf{g}^\# = 2 {}^0g^{01} \partial_0 \partial_1 + {}^0g^{ab} \partial_a \partial_b - \varepsilon \left\{ 2 {}^1g^{0a} \partial_0 \partial_a + {}^1g^{ab} \partial_a \partial_b \right\} \\ + \varepsilon^2 \left\{ {}^2h^{00} \partial_0^2 + 2 {}^2h^{0a} \partial_0 \partial_a + {}^2h^{ab} \partial_a \partial_b \right\} + O(\varepsilon^3),$$

with

$${}^2h^{00} = -({}^2g^{00} - {}^0g_{cd} {}^1g^{0c} {}^1g^{0d}), \\ {}^2h^{0a} = -({}^2g^{0a} - {}^0g_{cd} {}^1g^{0c} {}^1g^{ad}), \\ {}^2h^{ab} = -({}^2g^{ab} - {}^0g_{cd} {}^1g^{ac} {}^1g^{bd}).$$

In (B.11), the components g^{01} and g^{ab} are defined by

$$g^{01} g_{01} = 1, \quad g^{ac} g_{cb} = \delta_b^a.$$

Moreover, g^{01} is used to raise the sub-index 1, and g^{ab} is used to raise the sub-indices 2, 3. Thus, for example, we have

$$g^{100} = (g^{01})^2 g_{11}, \quad g^{20a} = g^{01} g^{ab} g_{1b}.$$

It follows from (B.11) that the transformed phase $u = x^0$ and the transformed transverse coordinates $y^a = x^a$ satisfy (B.1)–(B.2).

Appendix C: Nonzero connection coefficients and Ricci tensor components

In this appendix, we write out expressions for the nonzero connection coefficients and Ricci tensor components for the metric given in (3.7). These results follow by specialization of the expressions given in Appendix A to a metric of the form (3.7) with $u = x^0$ and $y^a = x^a$.

As before, we use the notation for derivatives in (3.6).

Nonzero connection coefficients at order ε^{-2} :

$$\Gamma^{-2}{}^0{}_{00} = g^{01} g_{01,\theta}, \quad \Gamma^{-2}{}^1{}_{ab} = -\frac{1}{2} g^{01} g_{ab,\theta}, \quad \Gamma^{-2}{}^a{}_{0b} = \frac{1}{2} g^{ac} g_{bc,\theta}.$$

Nonzero connection coefficients at order ε^{-1} :

$$\begin{aligned} \Gamma^{-1}{}^0{}_{0a} &= \frac{1}{2} g^{01} (g_{01,\bar{a}} + g_{1a,\theta}) - \frac{1}{2} g^{0b} g_{ab,\theta}, \\ \Gamma^{-1}{}^1{}_{1a} &= \frac{1}{2} g^{01} (g_{01,\bar{a}} - g_{1a,\theta}), \quad \Gamma^{-1}{}^1{}_{ab} = -\frac{1}{2} g^{01} g_{ab,\theta}, \\ \Gamma^{-1}{}^a{}_{01} &= -\frac{1}{2} g^{ac} (g_{01,\bar{c}} - g_{1c,\theta}), \quad \Gamma^{-1}{}^a{}_{0b} = \frac{1}{2} g^{ac} g_{bc,\theta} - \frac{1}{2} g^{ac} g_{bc,\theta}, \\ \Gamma^{-1}{}^a{}_{bc} &= \frac{1}{2} g^{ad} (g_{bd,\bar{c}} + g_{cd,\bar{b}} - g_{bc,\bar{d}}) + \frac{1}{2} g^{0a} g_{bc,\theta}. \end{aligned}$$

Nonzero connection coefficients at order ε^0 :

$$\begin{aligned}
\Gamma^0_{00} &= g^{01} g^0_{01,0}, & \Gamma^0_{01} &= \frac{1}{2} g^{0c} (g^0_{01,\bar{c}} - g^0_{1c,\theta}) + \frac{1}{2} g^{01} g^2_{11,\theta}, \\
\Gamma^0_{0a} &= \frac{1}{2} g^{01} (g^2_{1a,\theta} + g^0_{01,a}) - \frac{1}{2} g^{0c} g^1_{ac,\theta} + \frac{1}{2} h^{0c} g^0_{ac,\theta}, \\
\Gamma^0_{ab} &= \frac{1}{2} g^{01} (g^1_{1b,\bar{a}} + g^1_{1a,\bar{b}} - g^0_{ab,1}) \\
&\quad - \frac{1}{2} g^{0c} (g^0_{bc,\bar{a}} + g^0_{ac,\bar{b}} - g^0_{ab,\bar{c}}) - \frac{1}{2} h^{00} g^0_{ab,\theta}, \\
\Gamma^1_{11} &= \frac{1}{2} g^{01} (2 g^0_{01,1} - g^2_{11,\theta}), & \Gamma^1_{1a} &= \frac{1}{2} g^{01} (g^0_{01,a} - g^2_{1a,\theta}), \\
\Gamma^1_{ab} &= -\frac{1}{2} g^{01} (g^0_{ab,0} + g^2_{ab,\theta}), \\
\Gamma^a_{01} &= \frac{1}{2} g^{ac} (g^2_{1c,\theta} - g^0_{01,c}) - \frac{1}{2} g^{ac} (g^1_{1c,\theta} - g^0_{01,\bar{c}}), \\
\Gamma^a_{0b} &= \frac{1}{2} g^{ac} (g^2_{bc,\theta} + g^0_{bc,0}) - \frac{1}{2} g^{ac} g^1_{bc,\theta} + \frac{1}{2} h^{ac} g^0_{bc,\theta}, \\
\Gamma^a_{1b} &= \frac{1}{2} g^{ac} (g^0_{bc,1} + g^1_{1c,\bar{b}} - g^1_{1b,\bar{c}}) - \frac{1}{2} g^{0a} (g^0_{01,\bar{b}} - g^1_{1b,\theta}), \\
\Gamma^a_{bc} &= \frac{1}{2} g^{0a} g^1_{bc,\theta} - \frac{1}{2} h^{0a} g^0_{bc,\theta} + \frac{1}{2} g^{ad} (g^1_{bd,\bar{c}} + g^1_{cd,\bar{b}} - g^1_{bc,\bar{d}}) \\
&\quad - \frac{1}{2} g^{ad} (g^0_{bd,\bar{c}} + g^0_{cd,\bar{b}} - g^0_{bc,\bar{d}}) + \frac{1}{2} g^{ad} (g^0_{bd,c} + g^0_{cd,b} - g^0_{bc,d}).
\end{aligned}$$

Nonzero component of the Ricci curvature tensor at order ε^{-4} :

$$\begin{aligned}
(C.1) \quad R_{00} &= -\frac{1}{2} (g^{ab} g^0_{ab,\theta})_{,\theta} + \frac{1}{2} g^{01} g^0_{01,\theta} g^{ab} g^0_{ab,\theta} \\
&\quad - \frac{1}{4} g^{ac} g^0_{bc,\theta} g^{bd} g^0_{ad,\theta}.
\end{aligned}$$

Nonzero components of the Ricci curvature tensor at order ε^{-3} :

$$(C.2) \quad R_{00} = -\frac{1}{2} g^a_{a,\theta\theta} + \frac{1}{2} g^{01} g^0_{01,\theta} g^a_{a,\theta} - \frac{1}{2} g^{bd} g^0_{bc,\theta} g^c_{d,\theta},$$

$$\begin{aligned}
(C.3) \quad R_{0a} &= \frac{1}{2} (g_{ab} g^{01} g^b_{1,\theta})_{,\theta} + \frac{1}{4} g^{cd} g^0_{cd,\theta} g_{ab} g^{01} g^b_{1,\theta} + \frac{1}{2} (g^{bc} g^0_{ab,\theta})_{,\bar{c}} \\
&\quad - \frac{1}{2} (g^{01} g^0_{01,\bar{a}} + g^{cd} g^0_{cd,\bar{a}})_{,\theta} + \frac{1}{4} g^{bc} g^0_{ab,\theta} g^{de} g^0_{de,\bar{c}} \\
&\quad + \frac{1}{4} g^{01} g^0_{01,\bar{a}} g^{cd} g^0_{cd,\theta} - \frac{1}{4} g^{bd} g^0_{cd,\theta} g^{ce} g^0_{be,\bar{a}}.
\end{aligned}$$

Nonzero components of the Ricci curvature tensor at the order ε^{-2} :

$$R_{00}, R_{01}, R_{0a}, R_{ab}.$$

The components $\overset{-2}{R}_{00}$ and $\overset{-2}{R}_{0a}$ are the analogs of $\overset{-1}{R}_{00}$ and $\overset{-1}{R}_{0a}$ for the higher order components of the metric, and will not be listed here. The remaining components of the Ricci tensor are listed below:

$$(C.4) \quad \begin{aligned} \overset{-2}{R}_{01} = & - \left(\overset{0}{g}{}^{01} \overset{0}{g}_{01,1} + \frac{1}{2} \overset{0}{g}{}^{cd} \overset{0}{g}_{cd,1} \right)_{,\theta} - \frac{1}{4} \overset{0}{g}{}^{ab} \overset{0}{g}_{bc,\theta} \overset{0}{g}{}^{bd} \overset{0}{g}_{ad,1} \\ & + \frac{1}{2} \left(\overset{0}{g}_{01,\bar{a}} - \overset{1}{g}_{1a,\theta} \right) \overset{1}{g}{}^{0a} \Big|_{,\theta} - \frac{1}{2} \left(\overset{0}{g}_{01,\bar{a}} - \overset{1}{g}_{1a,\theta} \right) \overset{0}{g}{}^{ab} \Big|_{,\bar{b}} \\ & + \frac{1}{4} \left(\overset{0}{g}_{01,\bar{a}} - \overset{1}{g}_{1a,\theta} \right) \left(\overset{1}{g}{}^{0a} \overset{0}{g}{}^{cd} \overset{0}{g}_{cd,\theta} - \overset{0}{g}{}^{ab} \overset{0}{g}{}^{cd} \overset{0}{g}_{cd,\bar{b}} \right), \end{aligned}$$

$$(C.5) \quad \begin{aligned} \overset{-2}{R}_{ab} = & - \overset{0}{g}{}^{01} \overset{0}{g}_{ab,1\theta} + \frac{1}{2} \overset{0}{g}{}^{01} \overset{0}{g}{}^{cd} \left(\overset{0}{g}_{ac,\theta} \overset{0}{g}_{bd,1} + \overset{0}{g}_{ac,1} \overset{0}{g}_{bd,\theta} \right) \\ & - \frac{1}{4} \overset{0}{g}{}^{01} \overset{0}{g}{}^{cd} \left(\overset{0}{g}_{cd,1} \overset{0}{g}_{ab,\theta} + \overset{0}{g}_{cd,\theta} \overset{0}{g}_{ab,1} \right) + \overset{*}{R}_{ab} \\ & + \frac{1}{2} \left(\overset{1}{g}{}^{0c} \overset{0}{g}_{ab,\theta} \right)_{,\bar{c}} + \frac{1}{2} \overset{1}{g}{}^{0c} \overset{0}{g}_{ab,\theta} \left(\overset{0}{g}{}^{01} \overset{0}{g}_{01,\bar{c}} + \frac{1}{2} \overset{0}{g}{}^{de} \overset{0}{g}_{de,\bar{c}} \right) \\ & + \frac{1}{2} \left(\overset{1}{g}{}^{0c} \overset{0}{g}_{ab,\bar{c}} \right)_{,\theta} + \frac{1}{2} \overset{1}{g}{}^{0c} \overset{0}{g}_{ab,\bar{c}} \left(\overset{0}{g}{}^{01} \overset{0}{g}_{01,\theta} + \frac{1}{2} \overset{0}{g}{}^{de} \overset{0}{g}_{de,\theta} \right) \\ & + \frac{1}{2} \left(\overset{0}{g}{}^{01} \overset{0}{g}_{ac} \overset{1}{g}_{1,\bar{b}} \right)_{,\theta} + \frac{1}{2} \overset{0}{g}{}^{01} \overset{0}{g}_{ac} \overset{1}{g}_{1,\bar{b}} \left(\overset{0}{g}{}^{01} \overset{0}{g}_{01,\theta} + \frac{1}{2} \overset{0}{g}{}^{de} \overset{0}{g}_{de,\theta} \right) \\ & + \frac{1}{2} \left(\overset{0}{g}{}^{01} \overset{0}{g}_{bc} \overset{1}{g}_{1,\bar{a}} \right)_{,\theta} + \frac{1}{2} \overset{0}{g}{}^{01} \overset{0}{g}_{bc} \overset{1}{g}_{1,\bar{a}} \left(\overset{0}{g}{}^{01} \overset{0}{g}_{01,\theta} + \frac{1}{2} \overset{0}{g}{}^{de} \overset{0}{g}_{de,\theta} \right) \\ & - \frac{1}{2} \overset{0}{g}{}^{01} \overset{0}{g}{}^{cd} \overset{0}{g}_{ac,\theta} \overset{1}{g}_{1b,\bar{d}} + \frac{1}{2} \overset{0}{g}{}^{cd} \overset{1}{g}{}^{0e} \overset{0}{g}_{ac,\theta} \left(\overset{0}{g}_{be,\bar{d}} - \overset{0}{g}_{bd,\bar{e}} \right) \\ & - \frac{1}{2} \overset{0}{g}{}^{01} \overset{0}{g}{}^{cd} \overset{0}{g}_{bc,\theta} \overset{1}{g}_{1a,\bar{d}} + \frac{1}{2} \overset{0}{g}{}^{cd} \overset{1}{g}{}^{0e} \overset{0}{g}_{bc,\theta} \left(\overset{0}{g}_{ae,\bar{d}} - \overset{0}{g}_{ad,\bar{e}} \right) \\ & - \frac{1}{2} \left(\overset{1}{g}{}^0_c \overset{1}{g}{}^{0c} \overset{0}{g}_{ab,\theta} \right)_{,\theta} - \overset{1}{g}{}^0_c \overset{1}{g}{}^{0c} \overset{0}{g}_{ab,\theta} \left(\overset{0}{g}{}^{01} \overset{0}{g}_{01,\theta} + \frac{1}{2} \overset{0}{g}{}^{de} \overset{0}{g}_{de,\theta} \right) \\ & + \frac{1}{2} \overset{1}{g}{}^0_e \overset{1}{g}{}^{0e} \overset{0}{g}{}^{cd} \overset{0}{g}_{ac,\theta} \overset{0}{g}_{bd,\theta} - \frac{1}{2} \overset{0}{g}{}^{01} \overset{0}{g}_{ac} \overset{1}{g}_{1,\theta} \overset{0}{g}{}^{01} \overset{0}{g}_{bd} \overset{1}{g}_{1,\theta}. \end{aligned}$$

In (C.5),

$$\begin{aligned} \overset{*}{R}_{ab} = & \frac{1}{2} \left(\overset{0}{g}{}^{cd} \left(\overset{0}{g}_{bd,\bar{a}} + \overset{0}{g}_{ad,\bar{b}} - \overset{0}{g}_{ab,\bar{d}} \right) \right)_{,\bar{c}} \\ & + \frac{1}{2} \overset{0}{g}{}^{cd} \left(\overset{0}{g}_{bd,\bar{a}} + \overset{0}{g}_{ad,\bar{b}} - \overset{0}{g}_{ab,\bar{d}} \right) \left(\overset{0}{g}{}^{01} \overset{0}{g}_{01,\bar{c}} + \frac{1}{2} \overset{0}{g}{}^{ef} \overset{0}{g}_{ef,\bar{c}} \right) \\ & - \left(\overset{0}{g}{}^{01} \overset{0}{g}_{01,\bar{a}} + \frac{1}{2} \overset{0}{g}{}^{cd} \overset{0}{g}_{cd,\bar{a}} \right)_{,\bar{b}} - \frac{1}{2} \overset{0}{g}{}^{01} \overset{0}{g}_{01,\bar{a}} \overset{0}{g}{}^{01} \overset{0}{g}_{01,\bar{b}} \\ & - \frac{1}{4} \overset{0}{g}{}^{ce} \left(\overset{0}{g}_{de,\bar{a}} + \overset{0}{g}_{ae,\bar{d}} - \overset{0}{g}_{ad,\bar{e}} \right) \overset{0}{g}{}^{df} \left(\overset{0}{g}_{bf,\bar{c}} + \overset{0}{g}_{cf,\bar{b}} - \overset{0}{g}_{cb,\bar{f}} \right), \end{aligned}$$

denotes the components of the Ricci tensor of the zero order metric $g_{\alpha\beta}^0$ regarded as a function of the variables η^a only.

Appendix D: Variational principle

The variational principle for the vacuum Einstein field equations is

$$(D.1) \quad \delta S = 0, \quad S = \int L d^4x,$$

$$L = R \sqrt{-\det g},$$

where R is the scalar curvature,

$$R = g^{\alpha\beta} R_{\alpha\beta}.$$

Using (A.1) and (3.8) to expand the scalar curvature, we obtain

$$R = \frac{1}{\varepsilon^4} R^{-4} + \frac{1}{\varepsilon^3} R^{-3} + \frac{1}{\varepsilon^2} R^{-2} + O(\varepsilon^{-1}),$$

$$R^{-4} = g^{0\alpha\beta} R_{\alpha\beta}^{-4},$$

$$R^{-3} = g^{0\alpha\beta} R_{\alpha\beta}^{-3} - g^{1\alpha\beta} R_{\alpha\beta}^{-4},$$

$$R^{-2} = g^{0\alpha\beta} R_{\alpha\beta}^{-2} - g^{1\alpha\beta} R_{\alpha\beta}^{-3} + h^{2\alpha\beta} R_{\alpha\beta}^{-4}.$$

For a metric of the form (B.3), we find that

$$(D.2) \quad R^{-4} = 0,$$

$$R^{-3} = 0,$$

$$R^{-2} = 2 g^{001} R_{01}^{-2} + g^{0ab} R_{ab}^{-2} - 2 g^{10a} R_{0a}^{-3} + h^{200} R_{00}^{-4}.$$

We use (D.2) in (D.1) and expand the result with respect to ε . This gives

$$L = \frac{1}{\varepsilon^2} L^{(-2)} + O(\varepsilon^{-1}),$$

with

$$(D.3) \quad L^{(-2)} = \left\{ 2 R_{01}^{-2} + g_{01} g^{0ab} R_{ab}^{-2} - 2 g_1^a R_{0a}^{-3} + h_1^0 R_{00}^{-4} \right\} \sqrt{\det g_{ab}^0}.$$

We make the change of variables in the integration

$$d^4x = du dv dy dz = \varepsilon^3 d\theta dv d\eta dz,$$

and omit the integration with respect to the parametric variable z . The leading order asymptotic variational principle then becomes

$$(D.4) \quad \delta S^{(1)} = 0, \quad S^{(1)} = \int L^{(-2)} d\theta dv d\eta.$$

Variations of $S^{(1)}$ with respect to the second order metric component $\overset{2}{g}_{11}$ give the constraint (3.12). Variations with respect to the first order metric components $\overset{1}{g}_{0a}$ give the equation (3.13). Variations with respect to $\overset{0}{g}_{01}$ and $\overset{0}{g}_{ab}$ give the evolution equations (3.14).

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