Short-Time Existence for Scale-Invariant Hamiltonian Waves

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Abstract

We prove short-time existence of smooth solutions for a class of non-linear, and generally spatially nonlocal, Hamiltonian evolution equations that describe the self-interaction of weakly nonlinear scale-invariant waves. These equations include ones that describe weakly nonlinear hyperbolic surface waves, such as nonlinear Rayleigh waves in elasticity.

1 Introduction

In this paper we prove the short-time existence of solutions of a class of quadratically nonlinear evolution equations in one space-dimension. The equations are, in general, spatially nonlocal; an example is the singular integro-differential equation

$$\varphi_t + \frac{1}{2}\mathbf{H}\left[\psi^2\right]_{xx} + \psi\varphi_{xx} = 0,$$

where $\mathbf{H}$ denotes the Hilbert transform with respect to x and $\psi = \mathbf{H}[\varphi]$. According to the result we prove here, the initial value problem for this equation has a unique short-time solution in the Sobolev space H^s when $s > 5/2$.

This class of equations arises from a Hamiltonian description [33] of weakly nonlinear, translation-invariant, scale-invariant waves [2]. By 'scale-invariant' we mean that the wave motion is invariant under space-time

rescalings $x \mapsto \eta x$, $t \mapsto \eta t$ for all $\eta > 0$. As a result, the wave motion does not depend on any intrinsic length or time scales, and any space-time parameters on which it does depend must have the dimension of velocity.[1] One consequence of this scale-invariance is that the wave motion is non-dispersive.

A specific physical motivation for studying this class of equations comes from quasi-linear hyperbolic surface waves, such as Rayleigh waves in elasticity.[2] Surface waves that satisfy half-space or planar interface problems for systems of conservation laws (without source terms) are scale-invariant because neither the PDE nor the geometry define an intrinsic length scale.

We refer to surface waves, like Rayleigh waves, that decay exponentially away from the boundary on which they propagate, as 'genuine' surface waves, in contrast to 'radiative' or 'leaky' surface waves that propagate bulk waves into the interior. Weakly nonlinear 'genuine' surface waves are described asymptotically by nonlocal equations, whereas purely 'radiative' surface waves are typically described by standard local PDEs (see [3] for an example). One of our main goals is to prove short-time existence for these nonlocal surface wave equations, and it is useful to put them in the context of a broader Hamiltonian framework.

These surface wave equations are of interest, in particular, because they describe the formation of singularities in quasi-linear hyperbolic waves on the boundary of a half-space, a mechanism that differs from the familiar one of shock formation in the interior of a domain. Their study should also shed light on the nonlinear well-posedness of IBVPs for hyperbolic PDEs, and on the stability of shocks and other interfaces, when the uniform Lopatinski condition fails as a result of the presence of 'genuine' surface waves. (See [1], [5] for further discussion, and [29] for a local existence result for nonlinear boundary value problems in elasticity that permit Rayleigh waves.)

The canonical spectral form of the wave equation we study here is given in (2.3). This equation describes the spectral evolution of a wavefield subject to three-wave resonant interactions between wavenumbers $\{k, m, n\}$ such that $k + m + n = 0$. These three-wave resonances always occur for scale-invariant waves, since they are nondispersive. Equation (2.3) depends on a given kernel, or interaction coefficients, $T : \mathbb{Z}^3 \to \mathbb{C}$. Provided that $T(k, m, n)$ is not identically zero when $k + m + n = 0$, we expect that the

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[1]Wave motions, such as gravity water waves on deep water, that depend only on an acceleration rather than a velocity are not scale-invariant in the sense we use the term here.

[2]The term 'surface wave' is also used to refer to water waves, whose properties are different from the non-dispersive surface waves we discuss here.

dominant weakly nonlinear effects are those given in (2.3); thus, this equation provides a general description of nonlinear, scale-invariant waves in one space dimension when higher-order resonances can be neglected.

The Hamiltonian formalism used to derive (2.3) does not imply that the resulting equation is well-posed. Short-time existence is not obvious, and suitable bounds on the interaction coefficients appear to be required. (See [13] for some examples of related nonlocal, but non-Hamiltonian, evolution equations that are ill-posed, and the discussion of compacton-type equations in Section 3.4.) Consequently, the result we prove here provides a useful validation of the corresponding equations as consistent models for the self-interaction of nonlinear waves.

Solutions of this equation typically form singularities in finite time, so that, in general, a short-time existence result for smooth solutions is the best one can hope for. Singularity formation and the global existence of appropriate weak solutions are interesting questions, but we do not address them here. Unlike local existence, global existence presumably depends on the specific kernel, making a general global existence theory unlikely.

In proving our existence result, we find it convenient to study a non-canonical form of (2.3), given in (2.9), which may be regarded as a nonlocal generalization of the inviscid Burgers equation. The proof uses a standard Galerkin procedure, although we work in Fourier space because of the spatially nonlocal structure of the evolution equation.

The Hamiltonian structure of (2.3) imposes certain reality and symmetry conditions on the kernel T, given in (2.4). Scale-invariance further implies that T is a homogeneous function whose degree ν is a parameter that characterizes the space-time scaling invariance of the equation. This parameter takes the value $3/2$ for bulk waves and 2 for surface waves, but our analysis applies to kernels of arbitrary degree.

It is convenient to treat the cases $\nu \geq 3/2$ and $\nu < 3/2$ separately. For $\nu \geq 3/2$, which is our main case of interest, we assume that the interaction coefficients satisfy the bound in (4.9). This condition appears to be necessary to develop a local H^s-existence theory, and it is satisfied by the kernels that arise in surface wave problems. Our existence result is stated in Theorem 1, and the main estimate is given in Proposition 2. One quantity of interest is the number of square-integrable spatial derivatives required to prove short-time existence. As we show, short-time existence holds in H^s for the nonconservative equation (2.9) when $s > \nu$. An existence result for $\nu < 3/2$, under the assumption that the interaction coefficients satisfy the bound in (4.30), is given in Theorem 2.

In Section 2, we write out the general equation in various forms (canon-

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ical and noncanonical, spatial and spectral). In Section 3, we give some examples that arise from specific choices of the kernel T. These include the inviscid Burgers equation, the Hunter-Saxton equation, asymptotic equations for nonlinear surface waves, and equations related to compacton equations. In Section 4, we prove the local existence result, and in Section 5 we state an inequality used in the proof.

2 The evolution equation

We consider an equation for a scalar-valued function $\varphi(x, t)$ of the form

$$\varphi_t + \mathbf{H}\left[b(\varphi, \varphi)\right] = 0. \tag{2.1}$$

Here, $\mathbf{H}$ denotes the Hilbert transform with respect to the one-dimensional spatial variable x, and $b(\varphi, \varphi)$ is a quadratically nonlinear function. The skew-adjoint Hilbert transform plays the role of a Hamiltonian operator, and $b(\varphi, \varphi)$ is the variational derivative of a cubically nonlinear Hamiltonian functional.

The specific structure of the equation is most easily described in Fourier space. We study spatially periodic solutions for definiteness, so that

$$\varphi : \mathbb{T} \times I \to \mathbb{R}$$

where $\mathbb{T}$ is the circle of length 2π, and $I = (-t_*, t_*)$ is a time interval. Most of the results extend in a straightforward way to equations defined on $\mathbb{R}$ instead of $\mathbb{T}$.

We denote the Fourier coefficients of φ by $\widehat{\varphi} : \mathbb{Z} \times I \to \mathbb{C}$, where

$$\varphi(x, t) = \sum_{k=-\infty}^{\infty} \widehat{\varphi}(k, t) e^{ikx},$$

$$\widehat{\varphi}(k, t) = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(x, t) e^{-ikx} \, dx.$$

The Hilbert transform is defined by

$$\mathbf{H}[\varphi](x, t) = -\frac{1}{2\pi} \text{p.v.} \int_{\mathbb{T}} \cot\left(\frac{x - y}{2}\right) \varphi(y, t) \, dy,$$

$$\widehat{\mathbf{H}[\varphi]}(k, t) = i \operatorname{sgn} k \, \widehat{\varphi}(k, t),$$

where

$$\operatorname{sgn} k = \begin{cases} 1 & k > 0, \\ 0 & k = 0, \\ -1 & k < 0. \end{cases}$$

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The bilinear form b is defined in Fourier space by

$$\widehat{b(\varphi, \psi)}(k, t) = \sum_{n=-\infty}^{\infty} T(-k, k-n, n)\widehat{\varphi}(k-n, t)\widehat{\psi}(n, t), \qquad (2.2)$$

where $T : \mathbb{Z}^3 \to \mathbb{C}$ is a given kernel. Only the values of $T(k, m, n)$ with $k + m + n = 0$ appear in (2.2), so we could omit one of the arguments, but it is convenient to include all of them to exhibit the Hamiltonian symmetry property stated below. It then follows from (2.1) and (2.2) that $\widehat{\varphi}(k, t)$ satisfies the evolution equation

$$\widehat{\varphi}_t(k, t) + i\operatorname{sgn} k \sum_{n=-\infty}^{\infty} T(-k, k-n, n)\widehat{\varphi}(k-n, t)\widehat{\varphi}(n, t) = 0. \qquad (2.3)$$

We assume that $T : \mathbb{Z}^3 \to \mathbb{C}$ satisfies the following reality, symmetry, Hamiltonian, and scaling conditions:

$$\begin{aligned}
T(k, m, n) &= T^*(-k, -m, -n), \\
T(k, m, n) &= T(k, n, m), \\
T(k, m, n) &= T(m, n, k), \\
T(\eta k, \eta m, \eta n) &= \eta^\nu T(k, m, n) \qquad \text{for all } \eta > 0.
\end{aligned} \qquad (2.4)$$

Here, a star denotes the complex-conjugate. The symmetry of $T(k, m, n)$ under cyclic permutations of $\{k, m, n\}$ gives (2.3) a Hamiltonian structure and plays a crucial role in our local existence proof. For simplicity, we also assume that

$$T(0, m, n) = T(k, 0, n) = T(k, m, 0) = 0,$$

so that there are no mean-field interactions.

The homogeneity of T of degree ν corresponds to an invariance of (2.1) under rescalings

$$t \mapsto \eta t, \quad x \mapsto \eta x, \quad \varphi \mapsto \eta^{\nu-1}\varphi.$$

Dimensional analysis [2] shows that the parameter ν is related to the co-dimension $(n - d)$ of the waves by

$$\nu = \frac{n - d + 3}{2}. \qquad (2.5)$$

Here, d is the dimension of the wavenumber space, and n is the spatial dimension of the quantity that sets the mass dimension of the wave motion; for example, if this quantity is a density, then n is the dimension of the space in which the waves propagate. It follows that $\nu = 3/2$ for bulk waves with $d = n$, and $\nu = 2$ for surface waves with $d = n - 1$.

2.1 Hamiltonian structure

For simplicity, we consider functions whose spatial mean is zero, so that $\widehat{\varphi}(0,t) = 0$. (This involves no loss of generality, since $\widehat{\varphi}(0,t)$ is constant in time.) We define the Hamiltonian

$$
\mathcal{H}(\widehat{\varphi}, \widehat{\varphi}^*) = \sum_{m,n=1}^{\infty} \{ T(-m-n, m, n)\widehat{\varphi}^*(m+n, t)\widehat{\varphi}(m, t)\widehat{\varphi}(n, t)
$$
$$
+ T^*(-m-n, m, n)\widehat{\varphi}^*(m, t)\widehat{\varphi}^*(n, t)\widehat{\varphi}(m+n, t) \}.
$$

The complex canonical form of Hamilton's equations,

$$
i\widehat{\varphi}_t(k, t) = \frac{\delta\mathcal{H}}{\delta\widehat{\varphi}^*}(k, t) \qquad \text{for } k \in \mathbb{N},
$$

then gives

$$
i\widehat{\varphi}_t(k, t) = \sum_{n=1}^{k-1} T(-k, k-n, n)\widehat{\varphi}(k-n, t)\widehat{\varphi}(n, t)
$$
$$
+ 2\sum_{n=1}^{\infty} T^*(-k-n, k, n)\widehat{\varphi}(k+n, t)\widehat{\varphi}^*(n, t).
$$

The first term on the right-hand side of this equation describes the upward transfer of energy to higher wavenumbers, while the second term describes the reverse downward transfer. These equations may be rewritten as (2.3) for $k \in \mathbb{Z}$ by use of the properties of T and the fact that $\widehat{\varphi}^*(k, t) = \widehat{\varphi}(-k, t)$. Thus, equation (2.3) is Hamiltonian and $\{\widehat{\varphi}(k, t), \widehat{\varphi}^*(k, t) \mid k \in \mathbb{N}\}$ are complex canonical variables.

The Hamiltonian is conserved in time on smooth solutions, but it is an indefinite functional since it is cubically nonlinear. A positive-definite functional that is conserved on smooth solutions is the momentum

$$
\mathcal{P} = \sum_{k=1}^{\infty} k\widehat{\varphi}^*(k, t)\widehat{\varphi}(k) = \frac{1}{4\pi}\int_{\mathbb{T}} \left[|\partial_x|^{1/2}\,\varphi(x, t) \right]^2 dx. \tag{2.6}
$$

We do not know of other conserved functionals of (2.3) for general T, but the *a priori* estimate provided by the conservation of $\mathcal{P}$ (or the dissipation of $\mathcal{P}$ for weak solutions) is not sufficient to imply global existence.

2.2 Noncanonical variables

It is convenient to introduce noncanonical dependent variables

$$u : \mathbb{T} \times I \to \mathbb{R}, \qquad \widehat{u} : \mathbb{Z} \times I \to \mathbb{C}$$

defined by

$$u(x,t) = |\partial_x|^{1/2}\, \varphi(x,t), \qquad \widehat{u}(k,t) = |k|^{1/2}\widehat{\varphi}(k,t).$$

When rewritten in terms of $\widehat{u}$, equation (2.3) becomes

$$\widehat{u}_t(k,t) + ik \sum_{n=-\infty}^{\infty} S(-k, k-n, n)\widehat{u}(k-n,t)\widehat{u}(n,t) = 0, \qquad (2.7)$$

where

$$S(k,m,n) = \frac{T(k,m,n)}{|kmn|^{1/2}} \qquad \text{for } kmn \neq 0. \qquad (2.8)$$

We define $S(0,m,n) = S(k,0,n) = S(k,m,0) = 0$. The corresponding spatial form of (2.7) is then

$$u_t + a(u,u)_x = 0, \qquad (2.9)$$

where the bilinear form a is given by

$$\widehat{a(u,v)}(k,t) = \sum_{n=-\infty}^{\infty} S(-k, k-n, n)\widehat{u}(k-n,t)\widehat{v}(n,t). \qquad (2.10)$$

The form a in (2.9) is related to b in (2.1) by

$$b(\varphi, \psi) = |\partial_x|^{1/2}\, a\left(|\partial_x|^{1/2}\, \varphi, |\partial_x|^{1/2}\, \psi \right). \qquad (2.11)$$

Equation (2.9) is invariant under the rescalings

$$t \mapsto \eta t, \quad x \mapsto \eta x, \quad u \mapsto \eta^{\nu - 3/2} u.$$

As we will show in the next section, under suitable assumptions on S, the bilinear form $a(u,v)$ may be regarded as a kind of 'nonlocal product' of u and v, so that (2.9) is a nonlocal generalization of the inviscid Burgers equation. Our main result is that if the kernel $S(k,m,n)$ satisfies the bound in (4.8) below, where $\mu = \nu - 3/2 \geq 0$, then the initial value problem for (2.9)–(2.10) has a unique smooth solution for short times taking values in the Sobolev space $H^s(\mathbb{T})$ for $s > \nu$. Equivalently, equation (2.1)–(2.2) has a short-time H^s-solution for $s > \nu + 1/2$.

3 Examples

In this section, we give a some examples of equations that belong to the class described in the previous section. The simplest, and standard, example is the inviscid Burgers equation, but this class of equations is surprising rich, and includes other interesting PDEs and integro-differential equations. These examples also illustrate various different possibilities for local and global existence.

3.1 Burgers equation

Suppose that

$$T(k, m, n) = |kmn|^{1/2}. \tag{3.12}$$

Then $S(k, m, n) = 1$ for $kmn \neq 0$, and the noncanonical equation (2.9) corresponding to (3.12) is the inviscid Burgers equation

$$u_t + \left(u^2\right)_x = 0, \tag{3.13}$$

as can be seen from (2.10) by use of the convolution theorem. The kernel (3.12) is homogeneous of degree $3/2$, corresponding to $n = d$ in (2.5), and this equation describes bulk waves, such as weakly nonlinear sound waves.

Theorem 1 states local existence for (3.13) in H^s with $s > 3/2$, which is the usual short-time existence result for first-order quasilinear hyperbolic equations in one space dimension. Nontrivial periodic solutions form singularities in finite time, but the inviscid Burgers equation has global dissipative weak solutions that satisfy appropriate entropy conditions. These solutions are not, however, compatible with the Hamiltonian structure of (3.13).

3.2 Hunter-Saxton equation

Consider the kernel

$$T(k, m, n) = \frac{1}{2} |kmn|^{1/2} \left(\frac{1}{ik} + \frac{1}{im} + \frac{1}{in} \right) \quad \text{for } kmn \neq 0, \tag{3.14}$$

with $T(k, m, n) = 0$ when $kmn = 0$. Then

$$S(k, m, n) = \frac{1}{2} \left(\frac{1}{ik} + \frac{1}{im} + \frac{1}{in} \right) \quad \text{for } kmn \neq 0.$$

The corresponding noncanonical equation (2.9) is

$$u_t + \left(u \partial_x^{-1} u\right)_x - \frac{1}{2} \left(u^2 - \langle u^2 \rangle\right) = 0, \tag{3.15}$$

where ∂_x^{-1} is the anti-differentiation operator on zero-mean, periodic functions, and the angular brackets denote the spatial average over a period. It follows that $v = \partial_x^{-1}u$ satisfies the Hunter-Saxton equation [14],

$$v_t + vv_x - \frac{1}{2}\partial_x^{-1}\left(v_x^2 - \langle v_x^2 \rangle\right) = 0. \tag{3.16}$$

The kernel (3.14) has degree $\nu = 1/2$, corresponding to $n = d - 2$ in (2.5). This is consistent with the derivation of (3.16) in [14] for orientation waves in a massive director field. The waves propagate in three-dimensional space, so $d = 3$, and the mass parameter of the medium is set by the moment of inertia per unit volume of the director field, which has dimension M/L, so $n = 1$.

Theorem 2 (with $\lambda = 1$) implies the short-time existence of H^s-solutions of (3.15) for $s > 1/2$, and hence H^s-solutions of (3.16) for $s > 3/2$. This agrees with the short-time existence result of Yin [31]. Nontrivial periodic solutions of the Hunter-Saxton equation form singularities in finite time, and the equation has global weak solutions, including both dissipative and energy-conserving weak solutions ([6], [15], [34]). The conservative weak solutions are compatible with the Hamiltonian structure of the equation.

It is interesting to note that (3.16) is completely integrable ([4], [16]), and also arises as the high-frequency limit of the integrable Camassa-Holm equation [7],

$$v_t + 2\kappa v_x + 3vv_x = (v_t + vv_x)_{xx} - \frac{1}{2}\left(v_x^2\right)_x.$$

3.3 Surface wave equations

As discussed in the introduction, weakly nonlinear asymptotics for 'genuine' hyperbolic surface waves typically leads to nonlocal scale-invariant equations. These equations inherit a Hamiltonian structure from the Hamiltonian structure of the primitive physical equations (such as hyperelasticity or magnetohydrodynamics).

Perhaps the simplest kernel arising for surface waves is given by

$$T(k, m, n) = \frac{2|kmn|}{|k| + |m| + |n|}, \tag{3.17}$$

leading to the spectral evolution equation

$$\widehat{\varphi}_t(k, t) + i\,\mathrm{sgn}\,k \sum_{n=-\infty}^{\infty} \frac{2|k||k - n||n|}{|k| + |k - n| + |n|}\widehat{\varphi}(k - n, t)\widehat{\varphi}(n, t) = 0. \tag{3.18}$$

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This equation was proposed by Hamilton *et al* ([10], [11]) as a model equation for nonlinear Rayleigh waves, and it describes the propagation of surface waves on a tangential discontinuity in incompressible magnetohydrodynamics [1].

The kernel in (3.17) is homogeneous of degree $\nu = 2$. This degree is consistent with the result of dimensional analysis in (2.5), since the dimension d of the wavenumber space of surface waves that propagate along the boundary is one less that the spatial dimension n of the medium (whose density sets the mass parameter of the wave motion).

The spatial form of (3.18) may be written explicitly as ([2], [11])

$$\varphi_t + \frac{1}{2}\mathbf{H}\left[\psi^2\right]_{xx} + \psi\varphi_{xx} = 0, \tag{3.19}$$

where $\mathbf{H}$ is the Hilbert transform and $\psi = \mathbf{H}[\varphi]$. This equation corresponds to (2.1) with

$$
\begin{aligned}
b(\varphi, \varphi) &= \frac{1}{2}\left(\psi^2\right)_{xx} - \mathbf{H}\left[\psi\varphi_{xx}\right] \\
&= -\frac{\delta}{\delta\varphi}\int \psi\psi_x\varphi_x\, dx.
\end{aligned}
$$

The appearance of second-order spatial derivatives in the expression for b is misleading since some cancelation occurs. It follows from (2.11) and Proposition 1 (with $\mu = 1/2$) that $b : H^s \times H^s \to H^{s-1}$ involves the 'loss' of only one derivative when $s > 3/2$.

Theorem 1 implies that the noncanonical equation associated with (3.19) has short-time H^s-solutions when $s > 2$, so (3.19) has H^s-solutions when $s > 5/2$.

Numerical computations ([1], [11], [27]) show that solutions of (3.19) form singularities in which the derivative φ_x blows up, but φ appears to remain continuous. The global existence of appropriate weak solutions is an open question. As far as we have been able to determine, equation (3.19) is not completely integrable.

Similar equations describe the evolution of weakly nonlinear Rayleigh waves on an elastic half-space. In that case, $\varphi(x,t)$ corresponds to the horizontal displacement of the boundary of the half-space. For weakly nonlinear Rayleigh waves on an isotropic elastic half-space, $\widehat{\varphi}(k,t)$ satisfies an equation of the form (2.3) with the kernel (see [17], [18], [21], [24], [25], [32], for example)

$$T(k, n, m) = \frac{\alpha|k||m||n|}{|k| + r|m| + r|n|} + \frac{\alpha|k||m||n|}{r|k| + |m| + r|n|}$$

$$+\frac{\alpha|k||m||n|}{r|k|+r|m|+|n|}+\frac{\beta|k||m||n|}{r|k|+|m|+|n|}$$
$$+\frac{\beta|k||m||n|}{|n|+r|m|+|n|}+\frac{\beta|k||m||n|}{|k|+|m|+r|n|}$$
$$+\frac{\gamma|k||m||n|}{|k|+|m|+|n|}. \tag{3.20}$$

Here, $0 < r = c_t/c_l < 1$ is the ratio of the elastic solid's transverse and longitudinal wave speeds, c_t and c_l respectively, and α, β, γ are nonlinear elastic constants. The corresponding equation for $\varphi(x,t)$ has short-time H^s-solutions for $s > 5/2$.

More complicated, but qualitatively similar, kernels arise for nonlinear Rayleigh waves on a non-isotropic elastic half-space ([12], [22]). Kernels with a different degree of homogeneity ($\nu = 5/2$) arise in the weakly nonlinear description of elastic edge waves ([20], [26]) .

3.4 Compacton equations

The kernel
$$T(k,m,n) = i\,|kmn|^{1/2}\,kmn \tag{3.21}$$

leads to a noncanonical equation (2.9) that is a local PDE,

$$u_t + \left(u_x^2\right)_{xx} = 0.$$

Differentiating this equation with respect to x and setting $u_x = v$, we get

$$v_t + \left(v^2\right)_{xxx} = 0. \tag{3.22}$$

This equation is the high-frequency limit of the $K(2,2)$-equation introduced by Rosenau and Hyman [28]:

$$v_t + \left(v^2\right)_x + \left(v^2\right)_{xxx} = 0. \tag{3.23}$$

The $K(2,2)$-equation has 'compacton' solutions (that is, compactly supported traveling waves) given by

$$v(x,t) = \frac{4c}{3}\cos^2\left(\frac{x-ct}{4}\right) \qquad \text{if } |x-ct| \le 2\pi,$$

and $v(x,t) = 0$ if $|x - ct| > 2\pi$. Some numerical solutions of (3.23) are given in [9], [23].

Although (3.22) does not have compacton solutions, it has travelling wave solutions that describe the local non-smooth behavior of the compacton solutions of (3.23) near the 'edges' of the compacton; for example:

$$v(x,t) = \frac{c}{12}(x - ct)^2 \qquad \text{if } x \leq ct,$$

and $v(x,t) = 0$ if $x > ct$. The relationship between (3.22) and (3.23) is thus similar to the one between the Hunter-Saxton and Camassa-Holm equations, including the local description of the non-smooth behavior of the 'peakon' solutions of the Camassa-Holm equation by solutions of the Hunter-Saxton equation. As far as we know, however, neither (3.22) nor (3.23) is completely integrable.

An expansion of the spatial derivatives in (3.22),

$$v_t + 2vv_{xxx} + 6v_x v_{xx} = 0,$$

leads to a backward diffusion term when $v_x > 0$, which suggests that the equation may not be well-posed. The short-time existence proof given here does not apply to (3.22) since its kernel does not satisfy (4.9) below, and this condition seems to be necessary to obtain L^2-Sobolev estimates. It is conceivable that other estimates could lead to short-time existence for (3.22) or (3.23), but no such results appear to be known (although it is possible that a Cauchy-Kowalevski type argument could be used for analytic initial data [8]).

More generally, the kernel

$$T(k,m,n) = |kmn|^{1/2} (ikmn)^p$$

of degree $\nu = 3p + 3/2$, has the corresponding noncanonical equation

$$u_t + \partial_x^{p+1} (\partial_x^p u)^2 = 0,$$

which may be regarded as a higher-order generalization of the inviscid Burgers equation, to which it reduces when $p = 0$. Our short-time existence result does not apply to this equation when $p > 0$.

4   Local existence of smooth solutions

We consider an initial value problem for the noncanonical form of the equation described in Section 2:

$$u_t + a(u,u)_x = 0, \qquad\qquad (4.1)$$
$$u(x,0) = f(x).$$

Here, $u : \mathbb{T} \times I \to \mathbb{R}$, with I an open time interval containing the origin, $f : \mathbb{T} \to \mathbb{R}$ is given initial data, and the bilinear form a is defined by

$$\widehat{a(u,v)}(k,t) = \sum_{n=-\infty}^{\infty} S(-k, k-n, n) \widehat{u}(k-n, t) \widehat{v}(n, t). \qquad (4.2)$$

We assume that the kernel $S : \mathbb{Z}^3 \to \mathbb{C}$ satisfies

$$S(k, m, n) = 0 \quad \text{if } kmn = 0, \qquad\qquad\qquad (4.3)$$
$$S(k, m, n) = S^*(-k, -m, -n), \qquad\qquad\quad (4.4)$$
$$S(k, m, n) = S(k, n, m), \qquad\qquad\qquad\quad (4.5)$$
$$S(k, m, n) = S(m, n, k), \qquad\qquad\qquad\quad (4.6)$$
$$S(\eta k, \eta m, \eta n) = \eta^{\mu} S(k, m, n) \qquad \text{for } \eta > 0, \qquad (4.7)$$

These properties follow from those of T in (2.4) and the definition of S in (2.8), with

$$\mu = \nu - \frac{3}{2}.$$

We will focus our attention on kernels S with nonnegative degree $\mu \geq 0$. We consider kernels of negative degree in Section 4.4.

In order to develop a local existence theory, we assume that S satisfies the following condition for some constant C and all $k, m, n \in \mathbb{Z}$:

$$|S(k, m, n)| \leq C \min \left\{ |k|^{\mu}, |m|^{\mu}, |n|^{\mu} \right\}; \qquad\qquad (4.8)$$

equivalently, T satisfies

$$|T(k, m, n)| \leq C \, |kmn|^{1/2} \min \left\{ |k|^{\nu-3/2}, |m|^{\nu-3/2}, |n|^{\nu-3/2} \right\}. \qquad (4.9)$$

For example, the kernel (3.12) for the inviscid Burgers equation satisfies (4.9) with $\nu = 3/2$ and $C = 1$. The surface-wave kernel (3.17) satisfies (4.9) with $\nu = 2$ and $C = 1$, since

$$\frac{2|kmn|^{1/2}}{|k| + |m| + |n|} \leq \min \left\{ |k|^{1/2}, |m|^{1/2}, |n|^{1/2} \right\},$$

and the Rayleigh-wave kernel (3.20) satisfies (4.9) with $\nu = 2$ and

$$C = \frac{1}{2} \left(\frac{3|\alpha|}{r} + \frac{3|\beta|}{r} + |\gamma| \right).$$

By contrast, the compacton-type kernel (3.21) does not satisfy (4.9).

4.1   Notation

We denote by $\dot{H}^s(\mathbb{T})$, or $\dot{H}^s$ for short, the Hilbert space of periodic functions with zero mean and square integrable derivatives of the order s,

$$\dot{H}^s(\mathbb{T}) = \left\{ f : \mathbb{T} \to \mathbb{R} \mid \sum_{k=-\infty}^{\infty} |k|^{2s} \left| \widehat{f}(k) \right|^2 < \infty, \quad \widehat{f}(0) = 0 \right\}.$$

As inner product and norm on $\dot{H}^s$, we use

$$\langle f, g \rangle_s = \sum_{k=-\infty}^{\infty} |k|^{2s} \widehat{f}(k) \widehat{g}(-k),$$

$$\|f\|_s = \left(\sum_{k=-\infty}^{\infty} |k|^{2s} \left| \widehat{f}(k) \right|^2 \right)^{1/2}.$$

In particular,

$$\|f\|_0 = \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f(x)|^2 \, dx \right)^{1/2}.$$

For $\widehat{f} : \mathbb{Z} \to \mathbb{C}$, we also define

$$\left\| \widehat{f} \right\|_{\ell^p} = \left(\sum_{k=-\infty}^{\infty} \left| \widehat{f}(k) \right|^p \right)^{1/p}$$

so that

$$\|f\|_s = \left\| |k|^s \widehat{f} \right\|_{\ell^2}. \tag{4.10}$$

4.2   The bilinear form

First, we obtain some properties of the bilinear form a given by (4.2). In the simplest case when $S(k, m, n) = 1$ for $kmn \neq 0$, corresponding to $\mu = 0$, the form a is a multiplication operator on $\dot{H}^s$ for $s > 1/2$:

$$a(u, v) = uv - \frac{1}{2\pi} \int_{\mathbb{T}} uv \, dx. \tag{4.11}$$

As the following proposition shows, the form a has similar properties to a multiplication operator for any kernel S that satisfies (4.8), although we need some additional smoothness to ensure that $\dot{H}^s$ is an algebra under the product defined by a.

Proposition 1 *Let $a(u,v)$ be defined by (4.2), where $S : \mathbb{Z}^3 \to \mathbb{C}$ satisfies (4.3)–(4.8). Suppose that $s > \mu + 1/2$.*

(a) Then $a : \dot{H}^s \times \dot{H}^s \to \dot{H}^s$ is a bounded symmetric bilinear form.

(b) For all $u, v \in \dot{H}^{s+1}$,

$$a(u,v)_x = a(u_x, v) + a(u, v_x). \tag{4.12}$$

(c) For all $u, v, w \in \dot{H}^s$,

$$\frac{1}{2\pi} \int_{\mathbb{T}} u a(v,w)\, dx = \frac{1}{2\pi} \int_{\mathbb{T}} v a(w,u)\, dx, \tag{4.13}$$

and there is a constant $M_{s-\mu} > 0$ such that

$$\left| \frac{1}{2\pi} \int_{\mathbb{T}} u a(v,w)\, dx \right| \leq C M_{s-\mu} \|u\|_s \|v\|_0 \|w\|_0. \tag{4.14}$$

Proof. Using (4.2), we get

$$
\begin{aligned}
\|a(u,v)\|_s^2 &= \sum_k |k|^{2s} \left| \widehat{a(u,v)}(k) \right|^2 \\
&= \sum_k \left| |k|^s \sum_n S(-k, k-n, n) \widehat{u}(k-n) \widehat{v}(n) \right|^2. \tag{4.15}
\end{aligned}
$$

For $s > 0$ and $m, n \in \mathbb{R}$, we have

$$|m + n|^s \leq B_s \left(|m|^s + |n|^s \right),$$

where

$$B_s = \begin{cases} 1 & \text{if } 0 < s \leq 1, \\ 2^{s-1} & \text{if } s \geq 1. \end{cases}$$

Using this inequality, with $m = k - n$, and (4.8) in (4.15), we get

$$
\begin{aligned}
\|a(u,v)\|_s^2 &\leq B_s^2 \sum_k \left| \sum_n \left(|k-n|^s + |n|^s \right) |S(-k, k-n, n) \widehat{u}(k-n) \widehat{v}(n)| \right|^2 \\
&\leq B_s^2 C^2 \sum_k \left| \sum_n \left(|k-n|^s |n|^\mu + |k-n|^\mu |n|^s \right) |\widehat{u}(k-n) \widehat{v}(n)| \right|^2 \\
&\leq B_s^2 C^2 \left\| \left(|k|^s |\widehat{u}| \right) * \left(|k|^\mu |\widehat{v}| \right) + \left(|k|^\mu |\widehat{u}| \right) * \left(|k|^s |\widehat{v}| \right) \right\|_{\ell^2}^2.
\end{aligned}
$$

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Here, we denote the convolution of $\widehat{f}, \widehat{g} : \mathbb{Z} \to \mathbb{C}$ by

$$\widehat{f} * \widehat{g}(k) = \sum_{n=-\infty}^{\infty} \widehat{f}(k-n)\widehat{g}(n).$$

It follows that

$$\|a(u,v)\|_s \le B_s C \left\{ \left\| (|k|^s |\widehat{u}|) * (|k|^\mu |\widehat{v}|) \right\|_{\ell^2} + \left\| (|k|^\mu |\widehat{u}|) * (|k|^s |\widehat{v}|) \right\|_{\ell^2} \right\}.$$

Using Young's inequality,

$$\left\| \widehat{f} * \widehat{g} \right\|_{\ell^2} \le \left\| \widehat{f} \right\|_{\ell^1} \|\widehat{g}\|_{\ell^2}, \tag{4.16}$$

we get that

$$\|a(u,v)\|_s \le B_s C \left\{ \|u\|_s \left\| |k|^\mu |\widehat{v}| \right\|_{\ell^1} + \left\| |k|^\mu |\widehat{u}| \right\|_{\ell^1} \|v\|_s \right\}. \tag{4.17}$$

If $s > 1/2$, we have for all $f \in \dot{H}^s$ that

$$\left\| \widehat{f} \right\|_{\ell^1} \le M_s \|f\|_s, \tag{4.18}$$

where

$$M_s = \left(2 \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \right)^{1/2}. \tag{4.19}$$

Hence, if $s > \mu + 1/2$, inequalities (4.17) and (4.18) imply that

$$\|a(u,v)\|_s \le 2B_s C M_{s-\mu} \|u\|_s \|v\|_s.$$

Thus, a is bounded on $\dot{H}^s$. The symmetry and bilinearity of a is obvious.

To prove (4.12), we observe that

$$\begin{aligned} \widehat{a(u,v)_x}(k) &= ik\widehat{a(u,v)}(k) \\ &= ik \sum_n S(-k, k-n, n)\widehat{u}(k-n)\widehat{v}(n) \\ &= \sum_n i(k-n)S(-k, k-n, n)\widehat{u}(k-n)\widehat{v}(n) \\ &\quad + \sum_n inS(-k, k-n, n)\widehat{u}(k-n)\widehat{v}(n) \\ &= \widehat{a(u_x, v)}(k) + \widehat{a(u, v_x)}(k). \end{aligned}$$

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Equation (4.13) follows from the fact that

$$\frac{1}{2\pi}\int_{\mathbb{T}} ua(v,w)\,dx \;=\; \sum_{k}\widehat{u}(-k)\widehat{a(v,w)}(k)$$

$$=\; \sum_{k,n} S(-k,k-n,n)\widehat{u}(-k)\widehat{v}(k-n)\widehat{w}(n)$$

is symmetric under cyclic permutations of u,v,w. Moreover, using (4.8), the Cauchy-Schwartz inequality, Young's inequality, and (4.18) in this equation, we get for $s > \mu + 1/2$ that

$$\left|\frac{1}{2\pi}\int_{\mathbb{T}} ua(v,w)\,dx\right| \;\leq\; C\sum_{k,n} |k|^{\mu}\,|\widehat{u}(-k)\widehat{v}(k-n)\widehat{w}(n)|$$

$$\leq\; C\left\||\,(|k|^{\mu}|\widehat{u}|) * (|\widehat{v}|)\right\|_{\ell^2} \|\widehat{w}\|_{\ell^2}$$

$$\leq\; C\left\||k|^{\mu}\widehat{u}\right\|_{\ell^1} \|\widehat{v}\|_{\ell^2} \|\widehat{w}\|_{\ell^2}$$

$$\leq\; CM_{s-\mu}\,\|u\|_s\,\|v\|_0\,\|w\|_0 . \qquad \square$$

The form a satisfies other Sobolev inequalities. In Proposition 1, we prove only the ones that we need to use below.

4.3   Local existence

We define

$$K_s = CC_s M_{s-\mu-1}, \tag{4.20}$$

where C is the constant in (4.8), C_s is the constant in (5.2), and M_s is given by (4.19).

Theorem 1 *Suppose that $S : \mathbb{Z}^3 \to \mathbb{C}$ satisfies (4.3)–(4.7) and (4.8) with $\mu \geq 0$, the form a is given by (4.2), and $s > \mu + 3/2$. Then, for any $f \in \dot{H}^s(\mathbb{T})$, the initial-value problem (4.1) has a unique local solution*

$$u \in C\left(I; \dot{H}^s(\mathbb{T})\right) \cap C^1\left(I; \dot{H}^{s-1}(\mathbb{T})\right) \tag{4.21}$$

defined on the time interval $I = (-t_*, t_*)$, where

$$t_* = \frac{1}{K_s\,\|f\|_s}. \tag{4.22}$$

Proof. Let $P^N : \dot{H}^s(\mathbb{T}) \to \dot{H}^s(\mathbb{T})$ denote the orthogonal projection

$$P^N \left(\sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{ikx} \right) = \sum_{k=-N}^{N} \widehat{f}(k)e^{ikx}. \qquad (4.23)$$

We introduce the Galerkin approximations

$$u^N(x,t) = \sum_{k=-N}^{N} \widehat{u}^N(k,t)e^{ikx}$$

that satisfy the system of ODEs

$$\begin{aligned} u_t^N + P^N a\left(u^N, u^N\right)_x &= 0, \qquad (4.24)\\ u^N(x,0) &= P^N f(x). \end{aligned}$$

The main estimate for the local existence proof is given in the following Proposition.

Proposition 2 *Suppose $u^N(x,t)$ satisfies (4.24) and $s > \mu + 3/2$. Then*

$$\left| \frac{d}{dt} \left\| u^N \right\|_s \right| \le K_s \left\| u^N \right\|_s^2, \qquad (4.25)$$

where K_s is defined in (4.20).

Proof. Dropping the superscript N to simplify the notation, we compute from (4.24), (4.23), and (4.2) that

$$\frac{d}{dt} \sum_k |k|^{2s} \widehat{u}(k,t)\widehat{u}(-k,t)$$

$$+2i \sum_{k,n} k|k|^{2s} S(-k, k-n, n)\widehat{u}(-k,t)\widehat{u}(k-n,t)\widehat{u}(n,t) = 0. \quad (4.26)$$

Using the cyclic symmetry of $S(k,m,n)$, we can rewrite this equation as

$$\frac{d}{dt} \sum_k |k|^{2s} \left| \widehat{u}(k,t) \right|^2$$

$$+\frac{2i}{3} \sum_{k,n} \left\{ k|k|^{2s} - (k-n)|k-n|^{2s} - n|n|^{2s} \right\}$$

$$S(-k, k-n, n)\widehat{u}(-k,t)\widehat{u}(k-n,t)\widehat{u}(n,t) = 0.$$

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Using the inequality (5.2), we find that

$$\left| \frac{d}{dt} \sum_k |k|^{2s} |\widehat{u}(k,t)|^2 \right|$$

$$\leq \frac{2C_s}{3} \sum_{k,n} \{ |k|^s |k-n|^s |n| + |k-n|^s |n|^s |k| + |n|^s |k|^s |k-n| \}$$

$$|S(-k,k-n,n)|\ |\widehat{u}(-k,t)\widehat{u}(k-n,t)\widehat{u}(n,t)|. \quad (4.27)$$

Using (4.8) in the right hand side of (4.27), and applying the appropriate bound on each term, we get

$$\left| \frac{d}{dt} \sum_k |k|^{2s} |\widehat{u}(k,t)|^2 \right|$$

$$\leq \frac{2CC_s}{3} \sum_{k,n} \{ |k|^s |k-n|^s |n|^{\mu+1} + |k-n|^s |n|^s |k|^{\mu+1}$$

$$+ |n|^s |k|^s |k-n|^{\mu+1} \} |\widehat{u}(-k,t)\widehat{u}(k-n,t)\widehat{u}(n,t)|$$

$$\leq 2CC_s \sum_{k,n} |k|^s |k-n|^s |n|^{\mu+1} |\widehat{u}(-k,t)\widehat{u}(k-n,t)\widehat{u}(n,t)|.$$

Using the Cauchy-Schwartz inequality and Young's inequality (4.16) on the right hand side of this inequality, we get

$$\left| \frac{d}{dt} \left\| |k|^s \widehat{u} \right\|_{\ell^2}^2 \right| \leq 2CC_s \left\| |k|^s \widehat{u} \right\|_{\ell^2} \left\| (|k|^s|\widehat{u}|) * (|k|^{\mu+1}|\widehat{u}|) \right\|_{\ell^2},$$

$$\leq 2CC_s \left\| |k|^s \widehat{u} \right\|_{\ell^2}^2 \left\| |k|^{\mu+1}\widehat{u} \right\|_{\ell^1}.$$

It then follows from (4.10) and (4.18) that if $s > \mu + 3/2$, then

$$\left| \frac{d}{dt} \|u\|_s^2 \right| \leq 2CC_s M_{s-\mu-1} \|u\|_s^3.$$

Simplifying this equation and using (4.20) we get (4.25). $\square$

Using Gronwall's inequality, we deduce from (4.25) that

$$\left\| u^N \right\|_s (t) \leq \frac{\|f\|_s}{1 - K_s \|f\|_s |t|},$$

so the Galerkin approximations exist in $|t| < t_*$, where t_* is given by (4.22).

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It follows from Proposition 1 and standard arguments (for example, [30], §16.1, or [19]) that we can extract a subsequence (u^N) of Galerkin approximations such that

$$u^N \to u \in C_w \left(J; \dot{H}^s \right) \tag{4.28}$$

on any closed interval $J \subset I$, where u is a distributional solution of (4.1). (Here, C_w denotes the space of weakly continuous functions.) Proposition 2 and Gronwall's inequality imply that $t \mapsto \|u(t)\|_s$ is continuous. It follows that u is strongly $\dot{H}^s$-continuous, and thus satisfies (4.21) since a is strongly continuous on $\dot{H}^s$.

To prove uniqueness, we suppose that u, v are two solutions of (4.1) with the regularity stated in (4.21). Then, using integration by parts and the properties of a given in Proposition 1, we compute that

$$
\begin{aligned}
\frac{d}{dt} \|u - v\|_0^2 &= \frac{1}{2\pi} \frac{d}{dt} \int_{\mathbb{T}} (u - v)^2 \, dx \\
&= -\frac{1}{\pi} \int_{\mathbb{T}} (u - v) \left[a(u, u)_x - a(v, v)_x \right] dx \\
&= -\frac{1}{\pi} \int_{\mathbb{T}} (u - v) \, a(u + v, u - v)_x \, dx \\
&= \frac{1}{\pi} \int_{\mathbb{T}} (u - v)_x \, a(u + v, u - v) \, dx \\
&= \frac{1}{\pi} \int_{\mathbb{T}} (u + v) \, a \left(u - v, u_x - v_x \right) dx \\
&= \frac{1}{2\pi} \int_{\mathbb{T}} (u + v) \, a \left(u - v, u - v \right)_x \, dx \\
&= -\frac{1}{2\pi} \int_{\mathbb{T}} (u + v)_x \, a \left(u - v, u - v \right) dx.
\end{aligned}
$$

Since $s > \mu + 3/2$, it follows from this equation and (4.14) that

$$\left| \frac{d}{dt} \|u - v\|_0^2 \right| \leq C M_{s - \mu - 1} \|u + v\|_s \, \|u - v\|_0^2.$$

Hence, $u = v$ by Gronwall's inequality, and the solution is unique. $\square$

4.4 Negative degree kernels

In this section, we consider kernels S of negative degree $\mu = -\lambda < 0$ that satisfy the following condition for some constant C and all $kmn \neq 0$:

$$|S(k, m, n)| \leq \frac{C}{\min \left\{ |k|^\lambda, |m|^\lambda, |n|^\lambda \right\}}. \tag{4.29}$$

This corresponds to the assumption that T satisfies the condition

$$|T(k,m,n)| \leq \frac{C\,|kmn|^{1/2}}{\min\left\{|k|^{3/2-\nu}, |m|^{3/2-\nu}, |n|^{3/2-\nu}\right\}}. \tag{4.30}$$

where $\nu = 3/2 - \lambda < 3/2$. In particular, (4.30) is satisfied by the Hunter-Saxton kernel (3.14) with $\nu = 1/2$ and $C = 3/2$.

Let

$$K_{s,\lambda} = CC_{s,\lambda}M_{s+\lambda-1},$$

where C is the constant in (4.29), $C_{s,\lambda}$ is the constant in (5.1), and M_s is given by (4.19).

Theorem 2 *Suppose that $S : \mathbb{Z}^3 \to \mathbb{C}$ satisfies (4.3)–(4.7) and (4.29) with $\lambda > 0$, the form a is given by (4.2), and*

$$s > \max\left\{\frac{3}{2} - \lambda, \frac{1}{2}\right\}.$$

Then, for any $f \in \dot{H}^s(\mathbb{T})$, the initial-value problem (4.1) has a unique local solution

$$u \in C\left(I; \dot{H}^s(\mathbb{T})\right) \cap C^1\left(I; \dot{H}^{s-1}(\mathbb{T})\right).$$

defined on the time interval $I = (-t_*, t_*)$, where

$$t_* = \frac{1}{K_{s,\lambda}\|f\|_s}.$$

Proof. The proof is essentially the same as before. For $s > 1/2$, we get that $a : \dot{H}^s \times \dot{H}^s \to \dot{H}^s$ is a continuous bilinear form. From (4.26) and (4.29), the Galerkin approximations satisfy

$$\left|\frac{d}{dt}\sum_k |k|^{2s}\,|\widehat{u}(k,t)|^2\right|$$

$$\leq \frac{2}{3}\sum_{k,n}\left|k|k|^{2s} - (k-n)|k-n|^{2s} - n|n|^{2s}\right|$$

$$|S(-k, k-n, n)|\,|\widehat{u}(-k,t)\widehat{u}(k-n,t)\widehat{u}(n,t)|$$

$$\leq \frac{2C}{3}\sum_{k,n}\frac{\left|k|k|^{2s} - (k-n)|k-n|^{2s} - n|n|^{2s}\right|}{\min\left\{|k|^\lambda, |k-n|^\lambda, |n|^\lambda\right\}}\,|\widehat{u}(-k,t)\widehat{u}(k-n,t)\widehat{u}(n,t)|.$$

Using (5.1) in this equation, and summing over cyclic permutations, we find that

$$\left| \frac{d}{dt} \sum_k |k|^{2s} |\widehat{u}(k,t)|^2 \right| \leq 2CC_{s,\lambda} \sum_{k,n} |k|^s |k-n|^s |n|^{1-\lambda}$$
$$|\widehat{u}(-k,t)\widehat{u}(k-n,t)\widehat{u}(n,t)|$$
$$\leq 2CC_{s,\lambda} \left\| |k|^{1-\lambda} \widehat{u} \right\|_{\ell^1} \left\| |k|^s \widehat{u} \right\|_{\ell^2}^2.$$

Hence, using (4.18), with $s > 3/2 - \lambda$, and simplifying the result, we get

$$\left| \frac{d}{dt} \|u\|_s \right| \leq CC_{s,\lambda} M_{s+\lambda-1} \|u\|_s^2,$$

and the local existence result follows. □

5 An Inequality

For completeness, we prove an inequality used in our local existence proof.

Proposition 3 *For $s > 0$ and $\lambda \geq 0$, there exists a constant $C_{s,\lambda}$ such that for all nonzero $k, m, n \in \mathbb{R}$ with $k + m + n = 0$,*

$$\frac{\left| k|k|^{2s} + m|m|^{2s} + n|n|^{2s} \right|}{\min\left\{ |k|^\lambda, |m|^\lambda, |n|^\lambda \right\}}$$
$$\leq C_{s,\lambda} \left\{ |k|^s |m|^s |n|^{1-\lambda} + |n|^s |k|^s |m|^{1-\lambda} + |m|^s |n|^s |k|^{1-\lambda} \right\}. \quad (5.1)$$

Proof. We can assume without loss of generality that $0 < m \leq n$ and $k = -(m+n) < 0$. Writing $x = m/n$, we have

$$\frac{\left| k|k|^{2s} + m|m|^{2s} + n|n|^{2s} \right|}{|m|^\lambda \left\{ |k|^s |m|^s |n|^{1-\lambda} + |n|^s |k|^s |m|^{1-\lambda} + |m|^s |n|^s |k|^{1-\lambda} \right\}} = f_{s,\lambda}(x),$$

where $f_{s,\lambda} : (0, 1] \rightarrow (0, \infty)$ is given by

$$f_{s,\lambda}(x) = \frac{(x+1)^{2s+1} - x^{2s+1} - 1}{(x+1)^s (x^{s+\lambda} + x) + (x+1)^{1-\lambda} x^{s+\lambda}}.$$

This function is continuous on $(0, 1]$ and has a finite limit as $x \rightarrow 0^+$, with

$$f_{s,\lambda}\left(0^+\right) = \begin{cases} 0 & \text{if } 0 < s + \lambda < 1, \\ (2s+1)/3 & \text{if } s + \lambda = 1, \\ 2s + 1 & \text{if } s + \lambda > 1. \end{cases}$$

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Hence $f_{s,\lambda}$ is bounded on $(0,1]$ by $C_{s,\lambda}$, say, which proves (5.1). $\square$

In particular, if $\lambda = 0$, we write $f_s = f_{s,0}$ and $C_s = C_{s,0}$, so that

$$\left| k|k|^{2s} + m|m|^{2s} + n|n|^{2s} \right|$$
$$\leq C_s \left\{ |k|^s|m|^s|n| + |n|^s|k|^s|m| + |m|^s|n|^s|k| \right\} \qquad (5.2)$$

for all $k, m, n \in \mathbb{R}$ with $k + m + n = 0$. We have $f_1(x) = 1$, $f_3(x) = 7$, and $f_s(1) = 2^s - 1$. Numerical plots show that f_s is monotone decreasing on $(0,1]$ for $1 < s \leq 3$ and monotone increasing for $s \geq 3$, in which case

$$C_s = \left\{ \begin{array}{ll} 2s + 1 & \text{if } 1 < s \leq 3, \\ 2^s - 1 & \text{if } s \geq 3. \end{array} \right. \qquad (5.3)$$

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