Appendix

5.A. The Schwartz space and the Fourier transform

May the Schwartz be with you![3]

In this section, we summarize some results about Schwartz functions, tempered distributions, and the Fourier transform. For complete proofs, see [**16, 20**].

5.A.1. The Schwartz space. Since we will study the Fourier transform, we consider complex-valued functions.

Definition 5.51. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the topological vector space of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $f \in C^\infty(\mathbb{R}^n)$ and

$$x^\alpha \partial^\beta f(x) \to 0 \qquad \text{as } |x| \to \infty$$

for every pair of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. For $\alpha, \beta \in \mathbb{N}_0^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$ let

$$\|f\|_{\alpha,\beta} = \sup_{\mathbb{R}^n} \left| x^\alpha \partial^\beta f \right|. \tag{5.61}$$

A sequence of functions $\{f_k : k \in \mathbb{N}\}$ converges to a function f in $\mathcal{S}(\mathbb{R}^n)$ if

$$\|f_n - f\|_{\alpha,\beta} \to 0 \qquad \text{as } k \to \infty$$

for every $\alpha, \beta \in \mathbb{N}_0^n$.

That is, the Schwartz space consists of smooth functions whose derivatives (including the function itself) decay at infinity faster than any power; we say, for short, that Schwartz functions are rapidly decreasing. When there is no ambiguity, we will write $\mathcal{S}(\mathbb{R}^n)$ as $\mathcal{S}$.

Example 5.52. The function $f(x) = e^{-|x|^2}$ belongs to $\mathcal{S}(\mathbb{R}^n)$. More generally, if p is any polynomial, then $g(x) = p(x)\, e^{-|x|^2}$ belongs to $\mathcal{S}$.

Example 5.53. The function

$$f(x) = \frac{1}{(1 + |x|^2)^k}$$

does not belongs to $\mathcal{S}$ for any $k \in \mathbb{N}$ since $|x|^{2k} f(x)$ does not decay to zero as $|x| \to \infty$.

Example 5.54. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{-x^2} \sin\left(e^{x^2}\right)$$

does not belong to $\mathcal{S}(\mathbb{R})$ since $f'(x)$ does not decay to zero as $|x| \to \infty$.

The space $\mathcal{D}(\mathbb{R}^n)$ of smooth complex-valued functions with compact support is contained in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. If $f_k \to f$ in $\mathcal{D}$ (in the sense of Definition 3.7), then $f_k \to f$ in $\mathcal{S}$, so $\mathcal{D}$ is continuously imbedded in $\mathcal{S}$. Furthermore, if $f \in \mathcal{S}$, and $\eta \in C_c^\infty(\mathbb{R}^n)$ is a cutoff function with $\eta_k(x) = \eta(x/k)$, then $\eta_k f \to f$ in $\mathcal{S}$ as $k \to \infty$, so $\mathcal{D}$ is dense in $\mathcal{S}$.

[3] Spaceballs

The topology of $\mathcal{S}$ is defined by the countable family of semi-norms $\|\cdot\|_{\alpha,\beta}$ given in (5.61). This topology is not derived from a norm, but it is metrizable; for example, we can use as a metric

$$d(f,g) = \sum_{\alpha,\beta\in\mathbb{N}_0^n} \frac{c_{\alpha,\beta}\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}}$$

where the $c_{\alpha,\beta} > 0$ are any positive constants such that $\sum_{\alpha,\beta\in\mathbb{N}_0^n} c_{\alpha,\beta}$ converges. Moreover, $\mathcal{S}$ is complete with respect to this metric. A complete, metrizable topological vector space whose topology may be defined by a countable family of semi-norms is called a Fréchet space. Thus, $\mathcal{S}$ is a Fréchet space.

If we want to make explicit that a limit exists with respect to the Schwartz topology, we write

$$f = \mathcal{S}\text{-}\lim_{k\to\infty} f_k,$$

and call f the $\mathcal{S}$-limit of $\{f_k\}$.

If $f_k \to f$ as $k \to \infty$ in $\mathcal{S}$, then $\partial^\alpha f_k \to \partial^\alpha f$ for any multi-index $\alpha \in \mathbb{N}_0^n$. Thus, the differentiation operator $\partial^\alpha : \mathcal{S} \to \mathcal{S}$ is a continuous linear map on $\mathcal{S}$.

5.A.2. Tempered distributions. Tempered distributions are distributions (*c.f.* Section 3.3) that act continuously on Schwartz functions. Roughly speaking, we can think of tempered distributions as distributions that grow no faster than a polynomial at infinity.[4]

Definition 5.55. A tempered distribution T on $\mathbb{R}^n$ is a continuous linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. The topological vector space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{S}'$. If $\langle T, f\rangle$ denotes the value of $T \in \mathcal{S}'$ acting on $f \in \mathcal{S}$, then a sequence $\{T_k\}$ converges to T in $\mathcal{S}'$, written $T_k \rightharpoonup T$, if

$$\langle T_k, f\rangle \to \langle T, f\rangle \qquad \text{for every } f \in \mathcal{S}.$$

Since $\mathcal{D} \subset \mathcal{S}$ is densely and continuously imbedded, we have $\mathcal{S}' \subset \mathcal{D}'$. Moreover, a distribution $T \in \mathcal{D}'$ extends uniquely to a tempered distribution $T \in \mathcal{S}'$ if and only if it is continuous on $\mathcal{D}$ with respect to the topology on $\mathcal{S}$.

Every function $f \in L^1_{\mathrm{loc}}$ defines a regular distribution $T_f \in \mathcal{D}'$ by

$$\langle T_f, \phi\rangle = \int f\phi\,dx \qquad \text{for all } \phi \in \mathcal{D}.$$

If $|f| \le p$ is bounded by some polynomial p, then T_f extends to a tempered distribution $T_f \in \mathcal{S}'$, but this is not the case for functions f that grow too rapidly at infinity.

Example 5.56. The locally integrable function $f(x) = e^{|x|^2}$ defines a regular distribution $T_f \in \mathcal{D}'$ but this distribution does not extend to a tempered distribution.

Example 5.57. If $f(x) = e^x \cos(e^x)$, then $T_f \in \mathcal{D}'(\mathbb{R})$ extends to a tempered distribution $T \in \mathcal{S}'(\mathbb{R})$ even though the values of $f(x)$ grow exponentially as $x \to \infty$.

[4]The name 'tempered distribution' is short for 'distribution of temperate growth,' meaning polynomial growth.

This tempered distribution is the distributional derivative $T = T_g'$ of the regular distribution T_g where $f = g'$ and $g(x) = \sin(e^x)$:

$$\langle f, \phi\rangle = -\langle g, \phi'\rangle = -\int \sin(e^x)\phi(x)\,dx \qquad \text{for all } \phi \in \mathcal{S}.$$

The distribution T is decreasing in a weak sense at infinity because of the rapid oscillations of f.

EXAMPLE 5.58. The series

$$\sum_{n\in\mathbb{N}} \delta^{(n)}(x-n)$$

where $\delta^{(n)}$ is the nth derivative of the δ-function converges to a distribution in $\mathcal{D}'(\mathbb{R})$, but it does not converge in $\mathcal{S}'(\mathbb{R})$ or define a tempered distribution.

We define the derivative of tempered distributions in the same way as for distributions. If $\alpha \in \mathbb{N}_0^n$ is a multi-index, then

$$\langle \partial^\alpha T, \phi\rangle = (-1)^{|\alpha|}\langle T, \partial^\alpha \phi\rangle.$$

We say that a C^∞-function f is slowly growing if the function and all of its derivatives are of polynomial growth, meaning that for every $\alpha \in \mathbb{N}_0^n$ there exists a constant C_α and an integer N_α such that

$$|\partial^\alpha f(x)| \le C_\alpha \left(1 + |x|^2\right)^{N_\alpha}.$$

If f is C^∞ and slowly growing, then $f\phi \in \mathcal{S}$ whenever $\phi \in \mathcal{S}$, and multiplication by f is a continuous map on $\mathcal{S}$. Thus for $T \in \mathcal{S}'$, we may define the product $fT \in \mathcal{S}'$ by

$$\langle fT, \phi\rangle = \langle T, f\phi\rangle.$$

5.A.3. The Fourier transform on $\mathcal{S}$. The Schwartz space is a natural one to use for the Fourier transform. Differentiation and multiplication exchange rôles under the Fourier transform and therefore so do the properties of smoothness and rapid decrease. As a result, the Fourier transform is an automorphism of the Schwartz space. By duality, the Fourier transform is also an automorphism of the space of tempered distributions.

DEFINITION 5.59. The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is the function $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\hat{f}(k) = \frac{1}{(2\pi)^n}\int f(x)e^{-ik\cdot x}\,dx. \tag{5.62}$$

The inverse Fourier transform of f is the function $\check{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\check{f}(x) = \int f(k)e^{ik\cdot x}\,dk.$$

We generally use x to denote the variable on which a function f depends and k to denote the variable on which its Fourier transform depends.

EXAMPLE 5.60. For $\sigma > 0$, the Fourier transform of the Gaussian

$$f(x) = \frac{1}{(2\pi\sigma^2)^{n/2}}\,e^{-|x|^2/2\sigma^2}$$

is the Gaussian

$$\hat{f}(k) = \frac{1}{(2\pi)^n}\, e^{-\sigma^2|k|^2/2}$$

The Fourier transform maps differentiation to multiplication by a monomial and multiplication by a monomial to differentiation. As a result, $f \in \mathcal{S}$ if and only if $\hat{f} \in \mathcal{S}$, and $f_n \to f$ in $\mathcal{S}$ if and only if $\hat{f}_n \to \hat{f}$ in $\mathcal{S}$.

THEOREM 5.61. *The Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ defined by $\mathcal{F} : f \mapsto \hat{f}$ is a continuous, one-to-one map of $\mathcal{S}$ onto itself. The inverse $\mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S}$ is given by $\mathcal{F}^{-1} : f \mapsto \check{f}$. If $f \in \mathcal{S}$, then*

$$\mathcal{F}\left[\partial^\alpha f\right] = (ik)^\alpha \hat{f}, \qquad \mathcal{F}\left[(-ix)^\beta f\right] = \partial^\beta \hat{f}.$$

The Fourier transform maps the convolution product of two functions to the pointwise product of their transforms.

THEOREM 5.62. *If $f, g \in \mathcal{S}$, then the convolution $h = f * g \in \mathcal{S}$, and*

$$\hat{h} = (2\pi)^n \hat{f}\hat{g}.$$

If $f, g \in \mathcal{S}$, then

$$\int f\overline{g}\, dx = (2\pi)^n \int \hat{f}\overline{\hat{g}}\, dk.$$

In particular,

$$\int |f|^2\, dx = (2\pi)^n \int |\hat{f}|^2\, dk.$$

5.A.4. The Fourier transform on $\mathcal{S}'$. The main reason to introduce tempered distributions is that their Fourier transform is also a tempered distribution. If $\phi, \psi \in \mathcal{S}$, then by Fubini's theorem

$$\begin{aligned}
\int \phi\hat{\psi}\, dx &= \int \phi(x) \left[\frac{1}{(2\pi)^n} \int \psi(y) e^{-ix\cdot y}\, dy\right] dx \\
&= \int \left[\frac{1}{(2\pi)^n} \int \phi(x) e^{-ix\cdot y}\, dx\right] \psi(y)\, dy \\
&= \int \hat{\phi}\psi\, dx.
\end{aligned}$$

This motivates the following definition for the Fourier transform of a tempered distribution which is compatible with the one for Schwartz functions.

DEFINITION 5.63. If $T \in \mathcal{S}'$, then the Fourier transform $\hat{T} \in \mathcal{S}'$ is the distribution defined by

$$\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle \qquad \text{for all } \phi \in \mathcal{S}.$$

The inverse Fourier transform $\check{T} \in \mathcal{S}'$ is the distribution defined by

$$\langle \check{T}, \phi \rangle = \langle T, \check{\phi} \rangle \qquad \text{for all } \phi \in \mathcal{S}.$$

We also write $\hat{T} = \mathcal{F}T$ and $\check{T} = \mathcal{F}^{-1}T$. The linearity and continuity of the Fourier transform on $\mathcal{S}$ implies that $\hat{T}$ is a linear, continuous map on $\mathcal{S}$, so the Fourier transform of a tempered distribution is a tempered distribution. The invertibility of the Fourier transform on $\mathcal{S}$ implies that $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ is invertible with inverse $\mathcal{F}^{-1} : \mathcal{S}' \to \mathcal{S}'$.

EXAMPLE 5.64. If δ is the delta-function supported at 0, $\langle \delta, \phi\rangle = \phi(0)$, then

$$\langle \hat{\delta}, \phi\rangle = \langle \delta, \hat{\phi}\rangle = \hat{\phi}(0) = \frac{1}{(2\pi)^n}\int \phi(x)\, dx = \left\langle \frac{1}{(2\pi)^n}, \phi\right\rangle.$$

Thus, the Fourier transform of the δ-function is the constant function $(2\pi)^{-n}$. We may write this Fourier transform formally as

$$\delta(x) = \frac{1}{(2\pi)^n}\int e^{ik\cdot x}\, dk.$$

This result is consistent with Example 5.60. We have for the Gaussian δ-sequence that

$$\frac{1}{(2\pi\sigma^2)^{n/2}}\, e^{-|x|^2/2\sigma^2} \rightharpoonup \delta \qquad \text{in } \mathcal{S}' \text{ as } \sigma \to 0.$$

The corresponding Fourier transform of this limit is

$$\frac{1}{(2\pi)^n}\, e^{-\sigma^2|k|^2/2} \rightharpoonup \frac{1}{(2\pi)^n} \qquad \text{in } \mathcal{S}' \text{ as } \sigma \to 0.$$

If $T \in \mathcal{S}'$, it follows directly from the definitions and the properties of Schwartz functions that

$$\langle \widehat{\partial^\alpha T}, \phi\rangle = \langle \partial^\alpha T, \hat{\phi}\rangle = (-1)^{|\alpha|}\langle T, \partial^\alpha\hat{\phi}\rangle = \langle T, \widehat{(ik)^\alpha\phi}\rangle = \langle \hat{T}, (ik)^\alpha\phi\rangle = \langle (ik)^\alpha\hat{T}, \phi\rangle,$$

with a similar result for the inverse transform. Thus,

$$\widehat{\partial^\alpha T} = (ik)^\alpha\hat{T}, \qquad \widehat{(-ix)^\beta T} = \partial^\beta\hat{T}.$$

The Fourier transform does not define a map of the test function space $\mathcal{D}$ into itself, since the Fourier transform of a compactly supported function does not, in general, have compact support. Thus, the Fourier transform of a distribution $T \in \mathcal{D}'$ is not, in general, a distribution $\hat{T} \in \mathcal{D}'$; this explains why we define the Fourier transform for the smaller class of tempered distributions.

The Fourier transform maps the space $\mathcal{D}$ onto a space $\mathcal{Z}$ of real-analytic functions,[5] and one can define the Fourier transform of a general distribution $T \in \mathcal{D}'$ as an ultradistribution $\hat{T} \in \mathcal{Z}'$ acting on $\mathcal{Z}$. We will not consider this theory further here.

5.A.5. The Fourier transform on L^1.

If $f \in L^1(\mathbb{R}^n)$, then

$$\left|\int f(x)e^{-ik\cdot x}\, dx\right| \le \int |f|\, dx,$$

so we may define the Fourier transform $\hat{f}$ directly by the absolutely convergent integral in (5.62). Moreover,

$$\left|\hat{f}(k)\right| \le \frac{1}{(2\pi)^n}\int |f|\, dx.$$

It follows by approximation of f by Schwartz functions that $\hat{f}$ is a uniform limit of Schwartz functions, and therefore $\hat{f} \in C_0$ is a continuous function that approaches zero at infinity. We therefore get the following Riemann-Lebesgue lemma.

[5] A function $\phi : \mathbb{R} \to \mathbb{C}$ belongs to $\mathcal{Z}(\mathbb{R})$ if and only if it extends to an entire function $\phi : \mathbb{C} \to \mathbb{C}$ with the property that, writing $z = x+iy$, there exists $a > 0$ and for each $k = 0, 1, 2, \dots$ a constant C_k such that

$$\left|z^k\phi(z)\right| \le C_k e^{a|y|}.$$

Theorem 5.65. *The Fourier transform is a bounded linear map* $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ *and*

$$\left\| \hat{f} \right\|_{L^\infty} \leq \frac{1}{(2\pi)^n} \|f\|_{L^1}.$$

The range of the Fourier transform on L^1 is not all of C_0, however, and it is difficult to characterize.

5.A.6. The Fourier transform on L^2. The next theorem, called Parseval's theorem, states that the Fourier transform preserves the L^2-inner product and norm, up to factors of 2π. It follows that we may extend the Fourier transform by density and continuity from $\mathcal{S}$ to an isomorphism on L^2 with the same properties. Explicitly, if $f \in L^2$, we choose any sequence of functions $f_k \in \mathcal{S}$ such that f_k converges to f in L^2 as $k \to \infty$. Then we define $\hat{f}$ to be the L^2-limit of the $\hat{f}_k$. Note that it is necessary to use a somewhat indirect approach to define the Fourier transform on L^2, since the Fourier integral in (5.62) does not converge if $f \in L^2 \setminus L^1$.

Theorem 5.66. *The Fourier transform* $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is a one-to-one, onto bounded linear map. If* $f, g \in L^2(\mathbb{R}^n)$*, then*

$$\int f\overline{g}\, dx = (2\pi)^n \int \hat{f}\overline{\hat{g}}\, dk.$$

In particular,

$$\int |f|^2\, dx = (2\pi)^n \int |\hat{f}|^2\, dk.$$

5.A.7. The Fourier transform on L^p. The boundedness of the Fourier transform $\mathcal{F} : L^p \to L^{p'}$ for $1 < p < 2$ follows from its boundedness for $\mathcal{F} : L^1 \to L^\infty$ and $\mathcal{F} : L^2 \to L^2$ by use of the following Riesz-Thorin interpolation theorem.

Theorem 5.67. *Let* Ω *be a measure space and* $1 \leq p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$. *Suppose that*

$$T : L^{p_0}(\Omega) + L^{p_1}(\Omega) \to L^{q_0}(\Omega) + L^{q_1}(\Omega)$$

is a linear map such that $T : L^{p_i}(\Omega) \to L^{q_i}(\Omega)$ *for* $i = 0, 1$ *and*

$$\|Tf\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}}, \qquad \|Tf\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}}$$

for some constants M_0, M_1. *If* $0 < \theta < 1$ *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then $T : L^p(\Omega) \to L^q(\Omega)$ *maps* $L^p(\Omega)$ *into* $L^q(\Omega)$ *and*

$$\|Tf\|_{L^q} \leq M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}.$$

In this theorem, $L^{p_0}(\Omega) + L^{p_1}(\Omega)$ denotes the vector space of all complex-valued functions of the form $f = f_0 + f_1$ where $f_0 \in L^{p_0}(\Omega)$ and $f_1 \in L^{p_1}(\Omega)$. Note that if $q_0 = p_0'$ and $q_1 = p_1'$, then $q = p'$. An immediate consequence of this theorem and the L^1-L^2 estimates for the Fourier transform is the following Hausdorff-Young theorem.

THEOREM 5.68. *Suppose that $1 \le p \le 2$. The Fourier transform is a bounded linear map $\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ and*

$$\|\mathcal{F}f\|_{L^{p'}} \le \frac{1}{(2\pi)^n}\|f\|_{L^p}.$$

If $1 \le p < 2$, the range of the Fourier transform on L^p is not all of $L^{p'}$, and there exist functions $f \in L^{p'}$ whose inverse Fourier transform is a tempered distribution that is not regular. Correspondingly, if $p > 2$ the range of $\mathcal{F} : L^p \to \mathcal{S}'$ contains non-regular distributions. For example, $1 \in L^\infty$ and $\mathcal{F}(1) = \delta$.

5.A.8. The Sobolev spaces $H^s(\mathbb{R}^n)$. A function belongs to L^2 if and only if its Fourier transform belongs to L^2, and the Fourier transform preserves the L^2-norm. As a result, the Fourier transform provides a simple way to define L^2-Sobolev spaces on $\mathbb{R}^n$, including ones of fractional and negative order. This approach does not generalize to L^p-Sobolev spaces with $p \ne 2$, since there is no simple way to characterize when a function belongs to L^p in terms of its Fourier transform.

We define a function $\langle\cdot\rangle : \mathbb{R}^n \to \mathbb{R}$ by

$$\langle x\rangle = \left(1 + |x|^2\right)^{1/2}.$$

This function grows linearly at infinity, like $|x|$, but is bounded away from zero. (There should be no confusion with the use of angular brackets to denote a duality pairing.)

DEFINITION 5.69. For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ whose Fourier transform $\hat{f}$ is a regular distribution such that

$$\int \langle k\rangle^{2s}\left|\hat{f}(k)\right|^2\,dk < \infty.$$

The inner product and norm of $f, g \in H^s$ are defined by

$$(f,g)_{H^s} = (2\pi)^n \int \langle k\rangle^{2s}\hat{f}(k)\bar{\hat{g}}(k)\,dk, \qquad \|f\|_{H^s} = (2\pi)^n\left(\int \langle k\rangle^{2s}\left|\hat{f}(k)\right|^2\,dk\right)^{1/2}.$$

Thus, under the Fourier transform, $H^s(\mathbb{R}^n)$ is isomorphic to the weighted L^2-space

$$\hat{H}^s(\mathbb{R}^n) = \left\{f : \mathbb{R}^n \to \mathbb{C} : \langle k\rangle f \in L^2\right\}, \tag{5.63}$$

with inner product

$$\left(\hat{f},\hat{g}\right)_{\hat{H}^s} = (2\pi)^n \int \langle k\rangle^{2s}\hat{f}\bar{\hat{g}}\,dk.$$

The Sobolev spaces $\{H^s : s \in \mathbb{R}\}$ form a decreasing scale of Hilbert spaces with H^s continuously imbedded in H^r for $s > r$. If $s \in \mathbb{N}$ is a positive integer, then $H^s(\mathbb{R}^n)$ is the usual Sobolev space of functions whose weak derivatives of order less than or equal to s belong to $L^2(\mathbb{R}^n)$, so this notation is consistent with our previous notation.

We may give a spatial description of H^s for general $s \in \mathbb{R}$ in terms of the pseudo-differential operator $\Lambda : \mathcal{S}' \to \mathcal{S}'$ with symbol $\langle k\rangle$ defined by

$$\Lambda = (I - \Delta)^{1/2}, \qquad \widehat{(\Lambda f)}(k) = \langle k\rangle\hat{f}(k). \tag{5.64}$$

Then $f \in H^s$ if and only if $\Lambda^s f \in L^2$, and

$$(f, g)_{\hat{H}^s} = \int (\Lambda^s f)(\Lambda^s \bar{g})\, dx, \qquad \|f\|_{\hat{H}^s} = \left(\int |\Lambda^s f|^2\, dx\right)^{1/2}.$$

Thus, roughly speaking, a function belongs to H^s if it has s weak derivatives (or integrals if $s < 0$) that belong to L^2.

Example 5.70. If $\delta \in \mathcal{S}'(\mathbb{R}^n)$, then $\hat{\delta} = (2\pi)^{-n}$ and

$$\int \langle k \rangle^{2s} \hat{\delta}^2\, dk = \frac{1}{(2\pi)^{2n}} \int \langle k \rangle^{2s}\, dk$$

converges if $2s < -n$. Thus, $\delta \in H^s(\mathbb{R}^n)$ if $s < -n/2$, which is precisely when functions in H^s are continuous and pointwise evaluation at 0 is a bounded linear functional. More generally, every compactly supported distribution belongs to H^s for some $s \in \mathbb{R}$.

Example 5.71. The Fourier transform of $1 \in \mathcal{S}'$, given by $\hat{1} = \delta$, is not a regular distribution. Thus, $1 \notin H^s$ for any $s \in \mathbb{R}$.

We let

$$H^\infty = \bigcap_{s \in \mathbb{R}} H^s, \qquad H^{-\infty} = \bigcup_{s \in \mathbb{R}} H^s. \tag{5.65}$$

Then $\mathcal{S} \subset H^\infty \subset H^{-\infty} \subset \mathcal{S}'$ and by the Sobolev imbedding theorem $H^\infty \subset C_0^\infty$.

5.A.9. Riesz and Bessel potentials. For $0 < \alpha < n$, we define the Riesz potential $I_\alpha : \mathbb{R}^n \to \mathbb{R}$ by

$$I_\alpha(x) = \frac{1}{\gamma_\alpha} \frac{1}{|x|^{n-\alpha}}, \qquad \gamma_\alpha = \frac{2^\alpha \pi^{n/2} \Gamma(\alpha/2)}{\Gamma(n/2 - \alpha/2)}.$$

Since $\alpha > 0$, we have $I_\alpha \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

The Riesz potential of a function $\phi \in \mathcal{S}$ is defined by

$$I_\alpha * \phi(x) = \frac{1}{\gamma_\alpha} \int \frac{\phi(y)}{|x - y|^{n-\alpha}}\, dy.$$

The Fourier transform of this equation is

$$(\widehat{I_\alpha * \phi})(k) = \frac{1}{|k|^\alpha} \hat{\phi}(k).$$

Thus, we can interpret convolution with I_α as a homogeneous, spherically symmetric fractional integral operator of the order α. We write it symbolically as

$$I_\alpha * \phi = |D|^{-\alpha} \phi,$$

where $|D|$ is the operator with symbol $|k|$. In particular, for $\alpha = 2$ the potential I_2 is the Green's function of the Laplacian operator and $-\Delta(I_2 * \phi) = \phi$, which we may write symbolically as $I_2 * \phi = (-\Delta)^{-2}\phi = |D|^{-2}\phi$.

If we consider

$$|D|^{-\alpha} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

as a map from L^p to L^q, then a scaling argument similar to the one for the Sobolev imbedding theorem implies that the map can be bounded only if

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}. \tag{5.66}$$

The following Hardy-Littlewood-Sobolev inequality states that this map is, in fact, bounded for $1 < p < n/\alpha$. See [**18**] for a proof.

Theorem 5.72. *Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, and q is defined by (5.66). If $f \in L^p(\mathbb{R}^n)$, then $I_\alpha * f \in L^q(\mathbb{R}^n)$ and there exists a constant $C(n,\alpha,p)$ such that*

$$\|I_\alpha * f\|_{L^q} \leq C\,\|f\|_{L^p} \qquad \text{for every } f \in L^p(\mathbb{R}^n).$$

This inequality may be thought of as a generalization of the Gagliardo-Nirenberg inequality in Theorem 3.26 to fractional derivatives. If $\alpha = 1$, then $q = p^*$ is the Sobolev conjugate of p, and writing $f = |D|g$ we get

$$\|g\|_{L^{p^*}} \leq C\|f\|_{L^p}.$$

The Bessel potential corresponds to the operator

$$\Lambda^{-\alpha} = (I - \Delta)^{-\alpha/2} = \left(I + |D|^2\right)^{-\alpha/2}.$$

where Λ is defined in (5.64) and $\alpha > 0$. The operator $\Lambda^{-\alpha}$ is a non-homogeneous, spherically symmetric fractional integral operator; it plays an analogous role for non-homogeneous Sobolev spaces to the fractional derivative $|D|^{-\alpha}$ for homogeneous Sobolev spaces.

If $\phi \in \mathcal{S}$, then

$$(\widehat{\Lambda^{-\alpha}\phi})(k) = \frac{1}{\left(1 + |k|^2\right)^{\alpha/2}}\,\hat{\phi}(k).$$

Thus, by the convolution theorem,

$$\Lambda^{-\alpha}\phi = J_\alpha * \phi$$

where

$$J_\alpha = \mathcal{F}^{-1}\left[\frac{1}{\left(1 + |k|^2\right)^{\alpha/2}}\right].$$

This inverse transform does not have a simple explicit expression.