CHAPTER 6

Parabolic Equations

The theory of parabolic PDEs closely follows that of elliptic PDEs and, like elliptic PDEs, parabolic PDEs have strong smoothing properties. For example, there are parabolic versions of the maximum principle and Harnack's inequality, and a Schauder theory for Hölder continuous solutions [**23**]. Moreover, we may establish the existence and regularity of weak solutions of parabolic PDEs by the use of L^2-energy estimates.

6.1. The heat equation

Just as Laplace's equation is a prototypical example of an elliptic PDE, the heat equation

$$u_t = \Delta u + f \tag{6.1}$$

is a prototypical example of a parabolic PDE. This PDE has to be supplemented by suitable initial and boundary conditions to give a well-posed problem with a unique solution. As an example of such a problem, consider the following IBVP with Dirichlet BCs on a bounded open set $\Omega \subset \mathbb{R}^n$ for $u : \Omega \times [0, \infty) \to \mathbb{R}$:

$$\begin{aligned} &u_t = \Delta u + f(x,t) && \text{for } x \in \Omega \text{ and } t > 0,\\ &u(x,t) = 0 && \text{for } x \in \partial\Omega \text{ and } t > 0,\\ &u(x,0) = g(x) && \text{for } x \in \Omega. \end{aligned} \tag{6.2}$$

Here $f : \Omega \times (0, \infty) \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are a given forcing term and initial condition. This problem describes the evolution in time of the temperature $u(x,t)$ of a body occupying the region Ω containing a heat source f per unit volume, whose boundary is held at fixed zero temperature and whose initial temperature is g.

One important estimate (in L^∞) for solutions of (6.2) follows from the maximum principle. If $f \le 0$, corresponding to 'heat sinks,' then for any $T > 0$,

$$\max_{\overline{\Omega}\times[0,T]} u \le \max\left[0, \max_{\overline{\Omega}} g\right].$$

To derive this inequality, note that if u is a smooth function which attains a maximum at $x \in \Omega$ and $0 < t \le T$, then $u_t = 0$ if $0 < t < T$ or $u_t \ge 0$ if $t = T$ and $\Delta u \le 0$. Thus $u_t - \Delta u \ge 0$ which is impossible if $f < 0$, so u attains its maximum on $\partial\Omega \times [0,T]$, where $u = 0$, or at $t = 0$. The result for $f \le 0$ follows by a perturbation argument. The physical interpretation of this maximum principle in terms of thermal diffusion is that a local "hotspot" cannot develop spontaneously in the interior when no heat sources are present. Similarly, if $f \ge 0$, we have the minimum principle

$$\min_{\overline{\Omega}\times[0,T]} u \ge \min\left[0, \min_{\overline{\Omega}} g\right].$$

Another basic estimate for the heat equation (in L^2) follows from an integration of the equation. We multiply (6.1) by u, integrate over Ω, apply the divergence theorem, and use the BC that $u = 0$ on $\partial\Omega$ to obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega |Du|^2\,dx = \int_\Omega fu\,dx.$$

Integrating this equation with respect to time and using the initial condition, we get

$$\frac{1}{2}\int_\Omega u^2(x,t)\,dx + \int_0^t\int_\Omega |Du|^2\,dxds = \int_0^t\int_\Omega fu\,dxds + \frac{1}{2}\int_\Omega g^2\,dx. \tag{6.3}$$

For $0 \le t \le T$, we have from the Cauchy inequality with ϵ that

$$\begin{aligned}\int_0^t\int_\Omega fu\,dxds &\le \left(\int_0^t\int_\Omega f^2\,dxds\right)^{1/2}\left(\int_0^t\int_\Omega u^2\,dxds\right)^{1/2}\\ &\le \frac{1}{4\epsilon}\int_0^T\int_\Omega f^2\,dxds + \epsilon\int_0^T\int_\Omega u^2\,dxds\\ &\le \frac{1}{4\epsilon}\int_0^T\int_\Omega f^2\,dxds + \epsilon T\max_{0\le t\le T}\int_\Omega u^2\,dx.\end{aligned}$$

Thus, taking the supremum of (6.3) over $t \in [0,T]$ and using this inequality with $\epsilon T = 1/4$ in the result, we get

$$\frac{1}{4}\max_{[0,T]}\int_\Omega u^2(x,t)\,dx + \int_0^T\int_\Omega |Du|^2\,dxdt \le T\int_0^T\int_\Omega f^2\,dxdt + \frac{1}{2}\int_\Omega g^2\,dx.$$

It follows that we have an *a priori* energy estimate of the form

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H_0^1)} \le C\left(\|f\|_{L^2(0,T;L^2)} + \|g\|_{L^2}\right) \tag{6.4}$$

where $C = C(T)$ is a constant depending only on T. We will use this energy estimate to construct weak solutions.[1] The parabolic smoothing of the heat equation is evident from the fact that if $f = 0$, say, we can estimate not only the solution u but its derivative Du in terms of the initial data g.

6.2. General second-order parabolic PDEs

The qualitative properties of (6.1) are almost unchanged if we replace the Laplacian $-\Delta$ by any uniformly elliptic operator L on $\Omega\times(0,T)$. We write L in divergence form as

$$L = -\sum_{i,j=1}^n \partial_i\left(a^{ij}\partial_j u\right) + \sum_{j=1}^n b^j\partial_j u + cu \tag{6.5}$$

where $a^{ij}(x,t)$, $b^i(x,t)$, $c(x,t)$ are coefficient functions with $a^{ij} = a^{ji}$. We assume that there exists $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x,t)\xi_i\xi_j \ge \theta|\xi|^2 \qquad \text{for all } (x,t)\in\Omega\times(0,T) \text{ and } \xi\in\mathbb{R}^n. \tag{6.6}$$

[1] In fact, we will use a slightly better estimate in which $\|f\|_{L^2(0,T;L^2)}$ is replaced by the weaker norm $\|f\|_{L^2(0,T;H^{-1})}$.

The corresponding parabolic PDE is then

$$u_t + \sum_{j=1}^{n} b^j \partial_j u + cu = \sum_{i,j=1}^{n} \partial_i \left(a^{ij} \partial_j u\right) + f. \tag{6.7}$$

Equation (6.7) describes evolution of a temperature field u under the combined effects of diffusion a^{ij}, advection b^i, linear growth or decay c, and external heat sources f.

The corresponding IBVP with homogeneous Dirichlet BCs is

$$\begin{aligned} &u_t + Lu = f, \\ &u(x,t) = 0 \qquad\qquad \text{for } x \in \partial\Omega \text{ and } t > 0, \\ &u(x,0) = g(x) \qquad\quad \text{for } x \in \overline{\Omega}. \end{aligned} \tag{6.8}$$

Essentially the same estimates hold for this problem as for the heat equation. To begin with, we use the L^2-energy estimates to prove the existence of suitably defined weak solutions of (6.8).

6.3. Definition of weak solutions

To formulate a definition of a weak solution of (6.8), we first suppose that the domain Ω, the coefficients of L, and the solution u are smooth. Multiplying (6.7), by a test function $v \in C_c^\infty(\Omega)$, integrating the result over Ω, and applying the divergence theorem, we get

$$(u_t(t), v)_{L^2} + a\,(u(t), v; t) = (f(t), v)_{L^2} \qquad \text{for } 0 \le t \le T \tag{6.9}$$

where $(\cdot,\cdot)_{L^2}$ denotes the L^2-inner product

$$(u, v)_{L^2} = \int_\Omega u(x)v(x)\,dx,$$

and a is the bilinear form associated with L

$$\begin{aligned} a(u,v;t) = &\sum_{i,j=1}^{n} \int_\Omega a^{ij}(x,t)\partial_i u(x)\partial_j u(x)\,dx \\ &+ \sum_{j=1}^{n} \int_\Omega b^j(x,t)\partial_j u(x)v(x)\,dx + \int_\Omega c(x,t)u(x)v(x)\,dx. \end{aligned} \tag{6.10}$$

In (6.9), we have switched to the "vector-valued" viewpoint, and write $u(t) = u(\cdot, t)$.

To define weak solutions, we generalize (6.9) in a natural way. In order to ensure that the definition makes sense, we make the following assumptions.

Assumption 6.1. *The set $\Omega \subset \mathbb{R}^n$ is bounded and open, $T > 0$, and:*

1. *the coefficients of a in (6.10) satisfy $a^{ij}, b^j, c \in L^\infty(\Omega \times (0,T))$;*
2. *$a^{ij} = a^{ji}$ for $1 \le i,j \le n$ and the uniform ellipticity condition (6.6) holds for some constant $\theta > 0$;*
3. *$f \in L^2\left(0,T; H^{-1}(\Omega)\right)$ and $g \in L^2(\Omega)$.*

Here, we allow f to take values in $H^{-1}(\Omega) = H_0^1(\Omega)'$. We denote the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ by

$$\langle \cdot, \cdot \rangle : H^{-1}(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

Since the coefficients of a are uniformly bounded in time, it follows from Theorem 4.21 that

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \times (0,T) \to \mathbb{R}.$$

Moreover, there exist constants $C > 0$ and $\gamma \in \mathbb{R}$ such that for every $u, v \in H_0^1(\Omega)$

$$C\|u\|_{H_0^1}^2 \leq a(u,u;t) + \gamma\|u\|_{L^2}^2, \tag{6.11}$$

$$|a(u,v;t)| \leq C\,\|u\|_{H_0^1}\,\|v\|_{H_0^1}\,. \tag{6.12}$$

We then define weak solutions of (6.8) as follows.

Definition 6.2. A function $u : [0,T] \to H_0^1(\Omega)$ is a weak solution of (6.8) if:

(1) $u \in L^2\left(0,T; H_0^1(\Omega)\right)$ and $u_t \in L^2\left(0,T; H^{-1}(\Omega)\right)$;
(2) For every $v \in H_0^1(\Omega)$,

$$\langle u_t(t), v\rangle + a\left(u(t), v; t\right) = \langle f(t), v\rangle \tag{6.13}$$

for t pointwise *a.e.* in $[0,T]$ where a is defined in (6.10);
(3) $u(0) = g$.

The PDE is imposed in a weak sense by (6.13) and the boundary condition $u = 0$ on $\partial\Omega$ by the requirement that $u(t) \in H_0^1(\Omega)$. Two points about this definition deserve comment.

First, the time derivative u_t in (6.13) is to be understood as a distributional time derivative; that is $u_t = v$ if

$$\int_0^T \phi(t)u(t)\,dt = -\int_0^T \phi'(t)v(t)\,dt \tag{6.14}$$

for every $\phi : (0,T) \to \mathbb{R}$ with $\phi \in C_c^\infty(0,T)$. This is a direct generalization of the notion of the weak derivative of a real-valued function. The integrals in (6.14) are vector-valued Lebesgue integrals (Bochner integrals), which are defined in an analogous way to the Lebesgue integral of an integrable real-valued function as the L^1-limit of integrals of simple functions. See Section 6.A for further discussion of such integrals and the weak derivative of vector-valued functions. Equation (6.13) may then be understood in a distributional sense as an equation for the weak derivative u_t on $(0,T)$.

Second, it is not immediately obvious that the initial condition $u(0) = g$ in Definition 6.2 makes sense. We do not explicitly require any continuity on u, and since $u \in L^2\left(0,T; H_0^1(\Omega)\right)$ is defined only up to pointwise everywhere equivalence in $t \in [0,T]$ it is not clear that specifying a pointwise value at $t = 0$ imposes any restriction on u. As shown in Theorem 6.41, however, the conditions that $u \in L^2\left(0,T; H_0^1(\Omega)\right)$ and $u_t \in L^2\left(0,T; H^{-1}(\Omega)\right)$ imply that $u \in C\left([0,T]; L^2(\Omega)\right)$. Therefore, identifying u with its continuous representative, we see that the initial condition makes sense.

We then have the following existence result, whose proof will be given in the following sections.

Theorem 6.3. *Suppose that the conditions in Assumption 6.1 are satisfied. Then for every $f \in L^2\left(0,T; H^{-1}(\Omega)\right)$ and $g \in H_0^1(\Omega)$ there is a unique weak solution*

$$u \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)$$

of (6.8), in the sense of Definition 6.2, with $u_t \in L^2\left(0,T;H^{-1}(\Omega)\right)$*. Moreover, there is a constant* C*, depending only on* Ω*,* T*, and the coefficients of* L*, such that*

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H_0^1)} + \|u_t\|_{L^2(0,T;H^{-1})} \leq C\left(\|f\|_{L^2(0,T;H^{-1})} + \|g\|_{L^2}\right).$$

6.4. The Galerkin approximation

The basic idea of the existence proof is to approximate $u : [0,T] \to H_0^1(\Omega)$ by functions $u_N : [0,T] \to E_N$ that take values in a finite-dimensional subspace $E_N \subset H_0^1(\Omega)$ of dimension N. To obtain the u_N, we project the PDE onto E_N, meaning that we require that u_N satisfies the PDE up to a residual which is orthogonal to E_N. This gives a system of ODEs for u_N, which has a solution by standard ODE theory. Each u_N satisfies an energy estimate of the same form as the *a priori* estimate for solutions of the PDE. These estimates are uniform in N, which allows us to pass to the limit $N \to \infty$ and obtain a solution of the PDE.

In more detail, the existence of uniform bounds implies that the sequence $\{u_N\}$ is weakly compact in a suitable space and hence, by the Banach-Alaoglu theorem, there is a weakly convergent subsequence $\{u_{N_k}\}$ such that $u_{N_k} \rightharpoonup u$ as $k \to \infty$. Since the PDE and the approximating ODEs are linear, and linear functionals are continuous with respect to weak convergence, the weak limit of the solutions of the ODEs is a solution of the PDE. As with any similar compactness argument, we get existence but not uniqueness, since it is conceivable that different subsequences of approximate solutions could converge to different weak solutions. We can, however, prove uniqueness of a weak solution directly from the energy estimates. Once we know that the solution is unique, it follows by a compactness argument that we have weak convergence $u_N \rightharpoonup u$ of the full approximate sequence. One can then prove that the sequence, in fact, converges strongly in $L^2(0,T;H_0^1)$.

Methods such as this one, in which we approximate the solution of a PDE by the projection of the solution and the equation into finite dimensional subspaces, are called Galerkin methods. Such methods have close connections with the variational formulation of PDEs. For example, in the time-independent case of an elliptic PDE given by a variational principle, we may approximate the minimization problem for the PDE over an infinite-dimensional function space E by a minimization problem over a finite-dimensional subspace E_N. The corresponding equations for a critical point are a finite-dimensional approximation of the weak formulation of the original PDE. We may then show, under suitable assumptions, that as $N \to \infty$ solutions u_N of the finite-dimensional minimization problem approach a solution u of the original problem.

There is considerable flexibility the finite-dimensional spaces E_N one uses in a Galerkin method. For our analysis, we take

$$E_N = \langle w_1, w_2, \dots, w_N \rangle \tag{6.15}$$

to be the linear space spanned by the first N vectors in an orthonormal basis $\{w_k : k \in \mathbb{N}\}$ of $L^2(\Omega)$, which we may also assume to be an orthogonal basis of $H_0^1(\Omega)$. For definiteness, take the $w_k(x)$ to be the eigenfunctions of the Dirichlet Laplacian on Ω:

$$-\Delta w_k = \lambda_k w_k \quad w_k \in H_0^1(\Omega) \qquad \text{for } k \in \mathbb{N}. \tag{6.16}$$

From the previous existence theory for solutions of elliptic PDEs, the Dirichlet Laplacian on a bounded open set is a self-adjoint operator with compact resolvent, so that suitably normalized set of eigenfunctions have the required properties.

Explicitly, we have

$$\int_\Omega w_j w_k \, dx = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \qquad \int_\Omega Dw_j \cdot Dw_k \, dx = \begin{cases} \lambda_j & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

We may expand any $u \in L^2(\Omega)$ in an L^2-convergent series as

$$u(x) = \sum_{k \in \mathbb{N}} c^k w_k(x)$$

where $c^k = (u, w_k)_{L^2}$ and $u \in L^2(\Omega)$ if and only if

$$\sum_{k \in \mathbb{N}} |c_k|^2 < \infty.$$

Similarly, $u \in H_0^1(\Omega)$, and the series converges in $H_0^1(\Omega)$, if and only if

$$\sum_{k \in \mathbb{N}} \lambda_k |c_k|^2 < \infty.$$

We denote by $P_N : L^2(\Omega) \to E_N \subset L^2(\Omega)$ the orthogonal projection onto E_N defined by

$$P_N \left(\sum_{k \in \mathbb{N}} c^k w_k \right) = \sum_{k=1}^{N} c^k w_k. \tag{6.17}$$

We also denote by P_N the orthogonal projections $P_N : H_0^1(\Omega) \to E_N \subset H_0^1(\Omega)$ or $P_N : H^{-1}(\Omega) \to E_N \subset H^{-1}(\Omega)$, which we obtain by restricting or extending P_N from $L^2(\Omega)$ to $H_0^1(\Omega)$ or $H^{-1}(\Omega)$, respectively. Thus, P_N is defined on $H_0^1(\Omega)$ by (6.17) and on $H^{-1}(\Omega)$ by

$$\langle P_N u, v \rangle = \langle u, P_N v \rangle \qquad \text{for all } v \in H_0^1(\Omega).$$

While this choice of E_N is convenient for our existence proof, other choices are useful in different contexts. For example, the finite-element method is a numerical implementation of the Galerkin method which uses a space E_N of piecewise polynomial functions that are supported on simplices, or some other kind of element. Unlike the eigenfunctions of the Laplacian, finite-element basis functions, which are supported on a small number of adjacent elements, are straightforward to construct explicitly. Furthermore, one can approximate functions on domains with complicated geometry in terms of the finite-element basis functions by subdividing the domain into simplices, and one can refine the decomposition in regions where higher resolution is required. The finite-element basis functions are not exactly orthogonal, but they are almost orthogonal since they overlap only if they are supported on nearby elements. As a result, the associated Galerkin equations involve sparse matrices, which is crucial for their efficient numerical solution. One can obtain rigorous convergence proofs for finite-element methods that are similar to the proof discussed here (at least, if the underlying equations are not too complicated).

6.5. Existence of weak solutions

We proceed in three steps:

(1) Construction of approximate solutions;
(2) Derivation of energy estimates for approximate solutions;
(3) Convergence of approximate solutions to a solution.

After proving the existence of weak solutions, we will show that they are unique and make some brief comments on their regularity and continuous dependence on the data. We assume throughout this section, without further comment, that Assumption 6.1 holds.

6.5.1. Construction of approximate solutions. First, we define what we mean by an approximate solution. Let E_N be the N-dimensional subspace of $H_0^1(\Omega)$ given in (6.15)–(6.16) and P_N the orthogonal projection onto E_N given by (6.17).

Definition 6.4. A function $u_N : [0,T] \to E_N$ is an approximate solution of (6.8) if:

(1) $u_N \in L^2(0,T;E_N)$ and $u_{Nt} \in L^2(0,T;E_N)$;
(2) for every $v \in E_N$

$$(u_{Nt}(t), v)_{L^2} + a\left(u_N(t), v; t\right) = \langle f(t), v\rangle \tag{6.18}$$

pointwise *a.e.* in $t \in (0,T)$;
(3) $u_N(0) = P_N g$.

Since $u_N \in H^1(0,T;E_N)$, it follows from the Sobolev embedding theorem for functions of a single variable t that $u_N \in C([0,T];E_N)$, so the initial condition (3) makes sense. Condition (2) requires that u_N satisfies the weak formulation (6.13) of the PDE in which the test functions v are restricted to E_N. This is equivalent to the condition that

$$u_{Nt} + P_N L u_N = P_N f$$

for $t \in (0,T)$ pointwise *a.e.*, meaning that u_N takes values in E_N and satisfies the projection of the PDE onto E_N.[2]

To prove the existence of an approximate solution, we rewrite their definition explicitly as an IVP for an ODE. We expand

$$u_N(t) = \sum_{k=1}^{N} c_N^k(t) w_k \tag{6.19}$$

where the $c_N^k : [0,T] \to \mathbb{R}$ are absolutely continuous scalar coefficient functions. By linearity, it is sufficient to impose (6.18) for $v = w_1, \dots, w_N$. Thus, (6.19) is an approximate solution if and only if

$$c_N^k \in L^2(0,T), \quad c_{Nt}^k \in L^2(0,T) \qquad \text{for } 1 \le k \le N,$$

and $\{c_N^1, \dots, c_N^N\}$ satisfies the system of ODEs

$$c_{Nt}^j + \sum_{k=1}^{N} a^{jk} c_N^k = f^j, \quad c_N^j(0) = g^j \qquad \text{for } 1 \le j \le N \tag{6.20}$$

[2]More generally, one can define approximate solutions which take values in an N-dimensional space E_N and satisfy the projection of the PDE on another N-dimensional space F_N. This flexibility can be useful for problems that are highly non-self adjoint, but it is not needed here.

where

$$a^{jk}(t) = a(w_j, w_k; t), \quad f^j(t) = \langle f(t), w_j \rangle, \quad g^j = (g, w_j)_{L^2}.$$

Equation (6.20) may be written in vector form for $\vec{c} : [0,T] \to \mathbb{R}^N$ as

$$\vec{c}_{Nt} + A(t)\vec{c}_N = \vec{f}(t), \qquad \vec{c}_N(0) = \vec{g} \tag{6.21}$$

where

$$\vec{c}_N = \{c_N^1, \ldots, c_N^N\}^T, \quad \vec{f} = \{f^1, \ldots, f^N\}^T, \quad \vec{g} = \{g^1, \ldots, g^N\}^T,$$

and $A : [0,T] \to \mathbb{R}^{N\times N}$ is a matrix-valued function of t with coefficients $(a^{jk})_{j,k=1,N}$.

Proposition 6.5. *For every $N \in \mathbb{N}$, there exists a unique approximate solution $u_N : [0,T] \to E_N$ of (6.8).*

Proof. This result follows by standard ODE theory. We give the proof since the coefficient functions in (6.21) are bounded but not necessarily continuous functions of t. This is, however, sufficient since the ODE is linear.

From Assumption 6.1 and (6.12), we have

$$A \in L^\infty\left(0,T; \mathbb{R}^{N\times N}\right), \qquad \vec{f} \in L^2\left(0,T;\mathbb{R}^N\right). \tag{6.22}$$

Writing (6.21) as an equivalent integral equation, we get

$$\vec{c}_N = \Phi\left(\vec{c}_N\right), \qquad \Phi\left(\vec{c}_N\right)(t) = \vec{g} - \int_0^t A(s)\vec{c}_N(s)\,ds + \int_0^t \vec{f}(s)\,ds.$$

If follows from (6.22) that $\Phi : C\left([0,T_*];\mathbb{R}^N\right) \to C\left([0,T_*];\mathbb{R}^N\right)$ for any $0 < T_* \le T$. Moreover, if $\vec{p}, \vec{q} \in C\left([0,T_*];\mathbb{R}^N\right)$ then

$$\left\|\Phi\left(\vec{p}\right) - \Phi\left(\vec{q}\right)\right\|_{L^\infty([0,T_*];\mathbb{R}^N)} \le MT_* \left\|\vec{p} - \vec{q}\right\|_{L^\infty([0,T_*];\mathbb{R}^N)}$$

where

$$M = \sup_{0\le t\le T} \|A(t)\|.$$

Hence, if $MT_* < 1$, the map Φ is a contraction on $C\left([0,T_*];\mathbb{R}^N\right)$. The contraction mapping theorem then implies that there is a unique solution on $[0,T_*]$ which extends, after a finite number of applications of this result, to a solution $\vec{c}_N \in C\left([0,T];\mathbb{R}^N\right)$. The corresponding approximate solution satisfies $u_N \in C\left([0,T];E_N\right)$. Moreover,

$$\vec{c}_{Nt} = \Phi(\vec{c}_N)_t = -A\vec{c}_N + \vec{f} \in L^2\left(0,T;\mathbb{R}^N\right),$$

which implies that $u_{Nt} \in L^2\left(0,T;E_N\right)$. □

6.5.2. Energy estimates for approximate solutions. The derivation of energy estimates for the approximate solutions follows the derivation of the *a priori* estimate (6.4) for the heat equation. Instead of multiplying the heat equation by u, we take the test function $v = u_N$ in the Galerkin equations.

Proposition 6.6. *There exists a constant C, depending only on T, Ω, and the coefficient functions a^{ij}, b^j, c, such that for every $N \in \mathbb{N}$ the approximate solution u_N constructed in Proposition 6.5 satisfies*

$$\|u_N\|_{L^\infty(0,T;L^2)} + \|u_N\|_{L^2(0,T;H_0^1)} + \|u_{Nt}\|_{L^2(0,T;H^{-1})} \le C\left(\|f\|_{L^2(0,T;H^{-1})} + \|g\|_{L^2}\right).$$

PROOF. Taking $v = u_N(t) \in E_N$ in (6.18), we find that

$$(u_{Nt}(t), u_N(t))_{L^2} + a\left(u_N(t), u_N(t); t\right) = \langle f(t), u_N(t)\rangle$$

pointwise *a.e.* in $(0,T)$. Using this equation and the coercivity estimate (6.11), we find that there are constants $\beta > 0$ and $-\infty < \gamma < \infty$ such that

$$\frac{1}{2}\frac{d}{dt}\left\|u_N\right\|_{L^2}^2 + \beta\left\|u_N\right\|_{H_0^1}^2 \le \langle f, u_N\rangle + \gamma\left\|u_N\right\|_{L^2}^2$$

pointwise *a.e.* in $(0,T)$, which implies that

$$\frac{1}{2}\frac{d}{dt}\left(e^{-2\gamma t}\left\|u_N\right\|_{L^2}^2\right) + \beta e^{-2\gamma t}\left\|u_N\right\|_{H_0^1}^2 \le e^{-2\gamma t}\langle f, u_N\rangle.$$

Integrating this inequality with respect to t, using the initial condition $u_N(0) = P_N g$, and the projection inequality $\left\|P_N g\right\|_{L^2} \le \|g\|_{L^2}$, we get for $0 \le t \le T$ that

$$\frac{1}{2}e^{-2\gamma t}\left\|u_N(t)\right\|_{L^2}^2 + \beta\int_0^t e^{-2\gamma s}\left\|u_N\right\|_{H_0^1}^2\, ds \le \frac{1}{2}\|g\|_{L^2}^2 + \int_0^t e^{-2\gamma s}\langle f, u_N\rangle\, ds. \tag{6.23}$$

It follows from the definition of the H^{-1} norm, the Cauchy-Schwartz inequality, and Cauchy's inequality with ϵ that

$$\begin{aligned}
\int_0^t e^{-2\gamma s}\langle f, u_N\rangle\, ds &\le \int_0^t e^{-2\gamma s}\|f\|_{H^{-1}}\left\|u_N\right\|_{H_0^1}\, ds \\
&\le \left(\int_0^t e^{-2\gamma s}\|f\|_{H^{-1}}^2\, ds\right)^{1/2}\left(\int_0^t e^{-2\gamma s}\left\|u_N\right\|_{H_0^1}^2\, ds\right)^{1/2} \\
&\le C\|f\|_{L^2(0,T;H^{-1})}\left(\int_0^t e^{-2\gamma s}\left\|u_N\right\|_{H_0^1}^2\, ds\right)^{1/2} \\
&\le C\|f\|_{L^2(0,T;H^{-1})}^2 + \frac{\beta}{2}\int_0^t e^{-2\gamma s}\left\|u_N\right\|_{H_0^1}^2\, ds,
\end{aligned}$$

and using this result in (6.23) we get

$$\frac{1}{2}e^{-2\gamma t}\left\|u_N(t)\right\|_{L^2}^2 + \frac{\beta}{2}\int_0^t e^{-2\gamma s}\left\|u_N\right\|_{H_0^1}^2\, ds \le \frac{1}{2}\|g\|_{L^2}^2 + C\|f\|_{L^2(0,T;H^{-1})}^2.$$

Taking the supremum of this equation with respect to t over $[0,T]$, we find that there is a constant C such that

$$\left\|u_N\right\|_{L^\infty(0,T;L^2)}^2 + \left\|u_N\right\|_{L^2(0,T;H_0^1)}^2 \le C\left(\|g\|_{L^2}^2 + \|f\|_{L^2(0,T;H^{-1})}^2\right). \tag{6.24}$$

To estimate u_{Nt}, we note that since $u_{Nt}(t) \in E_N$

$$\left\|u_{Nt}(t)\right\|_{H^{-1}} = \sup_{v\in E_N\setminus\{0\}}\frac{(u_{Nt}(t), v)_{L^2}}{\|v\|_{H_0^1}}.$$

From (6.18) and (6.12) we have

$$\begin{aligned}
(u_{Nt}(t), v)_{L^2} &\le \left|a\left(u_N(t), v; t\right)\right| + \left|\langle f(t), v\rangle\right| \\
&\le C\left(\left\|u_N(t)\right\|_{H_0^1} + \|f(t)\|_{H^{-1}}\right)\|v\|_{H_0^1}
\end{aligned}$$

for every $v \in H_0^1$, and therefore

$$\left\|u_{Nt}(t)\right\|_{H^{-1}}^2 \le C\left(\left\|u_N(t)\right\|_{H_0^1}^2 + \|f(t)\|_{H^{-1}}^2\right).$$

Integrating this equation with respect to t and using (6.24) in the result, we obtain

$$\|u_{Nt}\|^2_{L^2(0,T;H^1_0)} \leq C\left(\|g\|^2_{L^2} + \|f\|^2_{L^2(0,T;H^{-1})}\right). \tag{6.25}$$

Equations (6.24) and (6.25) complete the proof. □

6.5.3. Convergence of approximate solutions. Next we prove that a subsequence of approximate solutions converges to a weak solution. We use a weak compactness argument, so we begin by describing explicitly the type of weak convergence involved.

We identify the dual space of $L^2\left(0,T;H^1_0(\Omega)\right)$ with $L^2\left(0,T;H^{-1}(\Omega)\right)$. The action of $f \in L^2\left(0,T;H^{-1}(\Omega)\right)$ on $u \in L^2\left(0,T;H^1_0(\Omega)\right)$ is given by

$$\langle\langle f, u\rangle\rangle = \int_0^T \langle f, u\rangle\, dt$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the duality pairing between $L^2\left(0,T;H^{-1}\right)$ and $L^2\left(0,T;H^1_0\right)$, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between H^{-1} and H^1_0.

Weak convergence $u_N \rightharpoonup u$ in $L^2\left(0,T;H^1_0(\Omega)\right)$ means that

$$\int_0^T \langle f(t), u_N(t)\rangle\, dt \to \int_0^T \langle f(t), u(t)\rangle\, dt \qquad \text{for every } f \in L^2\left(0,T;H^{-1}(\Omega)\right).$$

Similarly, $f_N \rightharpoonup f$ in $L^2\left(0,T;H^{-1}(\Omega)\right)$ means that

$$\int_0^T \langle f_N(t), u(t)\rangle\, dt \to \int_0^T \langle f(t), u(t)\rangle\, dt \qquad \text{for every } u \in L^2\left(0,T;H^1_0(\Omega)\right).$$

If $u_N \rightharpoonup u$ weakly in $L^2\left(0,T;H^1_0(\Omega)\right)$ and $f_N \to f$ strongly in $L^2\left(0,T;H^{-1}(\Omega)\right)$, or conversely, then $\langle f_N, u_N\rangle \to \langle f, u\rangle$.[3]

Proposition 6.7. *A subsequence of approximate solutions converges weakly in* $L^2\left(0,T;H^{-1}(\Omega)\right)$ *to a weak solution*

$$u \in C\left([0,T];L^2(\Omega)\right) \cap L^2\left(0,T;H^1_0(\Omega)\right)$$

of (6.8) with $u_t \in L^2\left(0,T;H^{-1}(\Omega)\right)$. *Moreover, there is a constant* C *such that*

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1_0)} + \|u_t\|_{L^2(0,T;H^{-1})} \leq C\left(\|f\|_{L^2(0,T;H^{-1})} + \|g\|_{L^2}\right).$$

Proof. Proposition 6.6 implies that the approximate solutions $\{u_N\}$ are bounded in $L^2\left(0,T;H^1_0(\Omega)\right)$ and their time derivatives $\{u_{Nt}\}$ are bounded in $L^2\left(0,T;H^{-1}(\Omega)\right)$. It follows from the Banach-Alaoglu theorem (Theorem 1.19) that we can extract a subsequence, which we still denote by $\{u_N\}$, such that

$$u_N \rightharpoonup u \quad \text{in } L^2\left(0,T;H^1_0\right), \qquad u_{Nt} \rightharpoonup u_t \quad \text{in } L^2\left(0,T;H^{-1}\right).$$

Let $\phi \in C^\infty_c(0,T)$ be a real-valued test function and $w \in E_M$ for some $M \in \mathbb{N}$. Taking $v = \phi(t)w$ in (6.18) and integrating the result with respect to t, we find that for $N \geq M$

$$\int_0^T \left\{(u_{Nt}(t), \phi(t)w)_{L^2} + a\left(u_N(t), \phi(t)w; t\right)\right\}\, dt = \int_0^T \langle f(t), \phi(t)w\rangle\, dt.$$

[3]It is, of course, *not* true that $f_N \rightharpoonup f$ and $u_N \rightharpoonup u$ implies $\langle f_N, u_N\rangle \to \langle f, u\rangle$. For example, $\sin N\pi x \rightharpoonup 0$ in $L^2(0,1)$ but $(\sin N\pi x, \sin N\pi x)_{L^2} \to 1/2$.

We take the limit of this equation as $N \to \infty$. Since the function $t \mapsto \phi(t)w$ belongs to $L^2(0,T;H_0^1)$, we have

$$\int_0^T (u_{Nt}, \phi w)_{L^2}\, dt = \langle\langle u_{Nt}, \phi w\rangle\rangle \to \langle\langle u_t, \phi w\rangle\rangle = \int_0^T \langle u_t, \phi w\rangle\, dt.$$

Moreover, the boundedness of a in (6.12) implies similarly that

$$\int_0^T a\left(u_N(t), \phi(t)w; t\right)\, dt \to \int_0^T a\left(u(t), \phi(t)w; t\right)\, dt.$$

It therefore follows that u satisfies

$$\int_0^T \phi\left[\langle u_t, w\rangle + a\left(u, w; t\right)\right]\, dt = \int_0^T \phi\langle f, w\rangle\, dt. \tag{6.26}$$

Since this holds for every $\phi \in C_c^\infty(0,T)$, we have

$$\langle u_t, w\rangle + a\left(u, w; t\right) = \langle f, w\rangle \tag{6.27}$$

pointwise $a.e.$ in $(0,T)$ for every $w \in E_M$. Moreover, since

$$\bigcup_{M\in\mathbb{N}} E_M$$

is dense in H_0^1, this equation holds for every $w \in H_0^1$, and therefore u satisfies (6.18).

Finally, to show that the limit satisfies the initial condition $u(0) = g$, we use the integration by parts formula (6.42) with $\phi \in C^\infty([0,T])$ such that $\phi(0) = 1$ and $\phi(T) = 0$ to get

$$\int_0^T \langle u_t, \phi w\rangle\, dt = \langle u(0), w\rangle - \int_0^T \phi_t\langle u, w\rangle.$$

Thus, using (6.27), we have

$$\langle u(0), w\rangle = \int_0^T \phi_t\langle u, w\rangle + \int_0^T \phi\left[\langle f, w\rangle - a\left(u, w; t\right)\right]\, dt.$$

Similarly, for the Galerkin appoximation with $w \in E_M$ and $N \geq M$, we get

$$\langle g, w\rangle = \int_0^T \phi_t\langle u_N, w\rangle + \int_0^T \phi\left[\langle f, w\rangle - a\left(u_N, w; t\right)\right]\, dt.$$

Taking the limit of this equation as $N \to \infty$, when the right-hand side converges to the right-hand side of the previosus equation, we find that $\langle u(0), w\rangle = \langle g, w\rangle$ for every $w \in E_M$, which implies that $u(0) = g$. □

6.5.4. Uniqueness of weak solutions. If u_1, u_2 are two solutions with the same data f, g, then by linearity $u = u_1 - u_2$ is a solution with zero data $f = 0$, $g = 0$. To show uniqueness, it is therefore sufficient to show that the only weak solution with zero data is $u = 0$.

Since $u(t) \in H_0^1(\Omega)$, we may take $v = u(t)$ as a test function in (6.13), with $f = 0$, to get

$$\langle u_t, u\rangle + a\left(u, u; t\right) = 0,$$

where this equation holds pointwise *a.e.* in $[0,T]$ in the sense of weak derivatives. Using (6.42) and the coercivity estimate (6.11), we find that there are constants $\beta > 0$ and $-\infty < \gamma < \infty$ such that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \beta\|u\|_{H_0^1}^2 \le \gamma\|u\|_{L^2}^2.$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \le \gamma\|u\|_{L^2}^2, \qquad u(0) = 0,$$

and since $\|u(0)\|_{L^2} = 0$, Gronwall's inequality implies that $\|u(t)\|_{L^2} = 0$ for all $t \ge 0$, so $u = 0$.

In a similar way, we get continuous dependence of weak solutions on the data. If u_i is the weak solution with data f_i, g_i for $i = 1, 2$, then there is a constant C independent of the data such that

$$\begin{aligned}&\|u_1 - u_2\|_{L^\infty(0,T;L^2)} + \|u_1 - u_2\|_{L^2(0,T;H_0^1)} \\ &\qquad \le C\left(\|f_1 - f_2\|_{L^2(0,T;H^{-1})} + \|g_1 - g_2\|_{L^2}\right).\end{aligned}$$

6.5.5. Regularity of weak solutions. For operators with smooth coefficients on smooth domains with smooth data f, g, one can obtain regularity results for weak solutions by deriving energy estimates for higher-order derivatives of the approximate Galerkin solutions u_N and taking the limit as $N \to \infty$. A repeated application of this procedure, and the Sobolev theorem, implies, from the Sobolev embedding theorem, that the weak solutions constructed above are smooth, classical solutions if the data satisfy appropriate compatibility relations. For a discussion of this regularity theory, see §7.1.3 of [**8**].

6.6. A semilinear heat equation

The Galerkin method is not restricted to linear or scalar equations. In this section, we briefly discuss its application to a semilinear heat equation. For more information and examples of the application of Galerkin methods to nonlinear evolutionary PDEs, see Temam [**35**].

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $T > 0$, and consider the semilinear, parabolic IBVP for $u(x,t)$

$$\begin{aligned} u_t &= \Delta u - f(u) && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= g(x) && \text{on } \Omega \times \{0\}. \end{aligned} \tag{6.28}$$

We suppose, for simplicity, that

$$f(u) = \sum_{k=0}^{2p-1} c_k u^k \tag{6.29}$$

is a polynomial of odd degree $2p - 1 \ge 1$. We also assume that the coefficient $c_{2p-1} > 0$ of the highest degree term is positive. We then have the following global existence result.

Theorem 6.8. *Let $T > 0$. For every $g \in L^2(\Omega)$, there is a unique weak solution*

$$u \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right) \cap L^{2p}\left(0,T; L^{2p}(\Omega)\right).$$

of (6.28)–(6.29).

The proof follows the standard Galerkin method for a parabolic PDE. We will not give it in detail, but we comment on the main new difficulty that arises as a result of the nonlinearity.

To obtain the basic *a priori* energy estimate, we multiplying the PDE by u,

$$\left(\frac{1}{2}u^2\right)_t + |Du|^2 + uf(u) = \operatorname{div}(uDu),$$

and integrate the result over Ω, using the divergence theorem and the boundary condition, which gives

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|Du\|_{L^2}^2 + \int_\Omega uf(u)\,dx = 0.$$

Since $uf(u)$ is an even polynomial of degree $2p$ with positive leading order coefficient, and the measure $|\Omega|$ is finite, there are constants $A > 0$, $C \geq 0$ such that

$$A\|u\|_{L^{2p}_{2p}} \leq \int_\Omega uf(u)\,dx + C.$$

We therefore have that

$$\frac{1}{2}\sup_{[0,T]}\|u\|_{L^2}^2 + \int_0^T \|Du\|_{L^2}^2\,dt + A\int_0^T \|u\|_{2p}^{2p}\,dt \leq CT + \frac{1}{2}\|g\|_{L^2}^2. \tag{6.30}$$

Note that if $\|u\|_{L^{2p}}$ is finite then $\|f(u)\|_{L^q}$ is finite for $q = (2p)'$, since then $q(2p-1) = 2p$ and

$$\int_\Omega |f(u)|^q\,dx \leq A\int_\Omega |u|^{q(2p-1)}\,dx + C \leq A\|u\|_{L^{2p}} + C.$$

Thus, in giving a weak formulation of the PDE, we want to use test functions

$$v \in H_0^1(\Omega) \cap L^{2p}(\Omega)$$

so that both $(Du, Dv)_{L^2}$ and $(f(u), v)_{L^2}$ are well-defined.

The Galerkin approximations $\{u_N\}$ take values in a finite dimensional subspace $E_N \subset H_0^1(\Omega) \cap L^{2p}(\Omega)$ and satisfy

$$u_{Nt} = \Delta u_N + P_N f(u_N),$$

where P_N is the orthogonal projection onto E_N in $L^2(\Omega)$. These approximations satisfy the same estimates as the *a priori* estimates in (6.30). The Galerkin ODEs have a unique local solution since the nonlinear terms are Lipschitz continuous functions of u_N. Moreover, in view of the *a priori* estimates, the local solutions remain bounded, and therefore they exist globally for $0 \leq t < \infty$.

Since the estimates (6.30) hold uniformly in N, we extract a subsequence that converges weakly (or weak-star) $u_N \rightharpoonup u$ in the appropriate topologies to a limiting function

$$u \in L^\infty\left(0, T; L^2\right) \cap L^2\left(0, T; H_0^1\right) \cap L^{2p}\left(0, T; L^{2p}\right).$$

Moreover, from the equation

$$u_t \in L^2\left(0, T; H^{-1}\right) + L^q\left(0, T; L^q\right)$$

where $q = (2p)'$ is the Hölder conjugate of $2p$.

In order to prove that u is a solution of the original PDE, however, we have to show that

$$f(u_N) \rightharpoonup f(u) \tag{6.31}$$

in an appropriate sense. This is not immediately clear because of the lack of weak continuity of nonlinear functions; in general, even if $f(u_N) \rightharpoonup \bar{f}$ converges, we may not have $\bar{f} = f(u)$. To show (6.31), we use the compactness Theorem 6.9 stated below. This theorem and the weak convergence properties found above imply that there is a subsequence of approximate solutions such that

$$u_N \to u \qquad \text{strongly in } L^2(0,T;L^2).$$

This is equivalent to strong-L^2 convergence on $\Omega \times (0,T)$. By the Riesz-Fischer theorem, we can therefore extract a subsequence so that $u_N(x,t) \to u(x,t)$ pointwise *a.e.* on $\Omega \times (0,T)$. Using the dominated convergence theorem and the uniform bounds on the approximate solutions, we find that for every $v \in H_0^1(\Omega) \cap L^{2p}(\Omega)$

$$(f(u_N(t)), v)_{L^2} \to (f(u(t)), v)_{L^2}$$

pointwise a.e. on $[0,T]$.

Finally, we state the compactness theorem used here.

Theorem 6.9. *Suppose that $X \hookrightarrow Y \hookrightarrow Z$ are Banach spaces, where X, Z are reflexive and X is compactly embedded in Y. Let $1 < p < \infty$. If the functions $u_N : (0,T) \to X$ are such that $\{u_N\}$ is uniformly bounded in $L^2(0,T;X)$ and $\{u_{Nt}\}$ is uniformly bounded in $L^p(0,T;Z)$, then there is a subsequence that converges strongly in $L^2(0,T;Y)$.*

The proof of this theorem is based on Ehrling's lemma.

Lemma 6.10. *Suppose that $X \hookrightarrow Y \hookrightarrow Z$ are Banach spaces, where X is compactly embedded in Y. For any $\epsilon > 0$ there exists a constant C_ϵ such that*

$$\|u\|_Y \leq \epsilon \|u\|_X + C_\epsilon \|u\|_Z .$$

Proof. If not, there exists $\epsilon > 0$ and a sequence $\{u_n\}$ in X with $\|u_n\|X = 1$ such that

$$\|u_n\|_Y > \epsilon \|u_n\|_X + n \|u_n\|_Z \tag{6.32}$$

for every $n \in \mathbb{N}$. Since $\{u_n\}$ is bounded in X and X is compactly embedded in Y, there is a subsequence, which we still denote by $\{u_n\}$ that converges strongly in Y, to u, say. Then $\{\|u_n\|_Y\}$ is bounded and therefore $u = 0$ from (6.32). However, (6.32) also implies that $\|u_n\|_Y > \epsilon$ for every $n \in \mathbb{N}$, which is a contradiction. □

If we do not impose a sign condition on the nonlinearity, then solutions may 'blow up' in finite time, as for the ODE $u_t = u^3$, and then we do not get global existence.

Example 6.11. Consider the following one-dimensional IBVP [**16**] for $u(x,t)$ in $0 < x < 1$, $t > 0$:

$$\begin{aligned} &u_t = u_{xx} + u^3, \\ &u(0,t) = u(1,t) = 0, \\ &u(x,0) = g(x). \end{aligned} \tag{6.33}$$

Suppose that $u(x,t)$ is smooth solution, and let

$$c(t) = \int_0^1 u(x,t) \sin(\pi x)\, dx$$

denote the first Fourier sine coefficient of u. Multiplying the PDE by $\sin(\pi x)$, integrating with respect to x over $(0,1)$, and using Green's formula to write

$$\begin{aligned}\int_0^1 u_{xx}(x,t)\sin(\pi x)\,dx &= \left[u_x \sin(\pi x) - \pi u \cos(\pi) x\right]_0^1 - \pi^2 \int_0^1 u(x,t)\sin(\pi x)\,dx \\ &= -\pi^2 c,\end{aligned}$$

we get that

$$\frac{dc}{dt} = -\pi^2 c + \int_0^1 u^3 \sin(\pi x)\,dx.$$

Now suppose that $g(x) \geq 0$. Then the maximum principle implies that $u(x,t) \geq 0$ for all $0 < x < 1$, $t > 0$. It then follows from Hölder inequality that

$$\begin{aligned}\int_0^1 u \sin(\pi x)\,dx &= \int_0^1 \left[u^3 \sin(\pi x)\right]^{1/3} \left[\sin(\pi x)\right]^{2/3}\,dx \\ &\leq \left(\int_0^1 u^3 \sin(\pi x)\,dx\right)^{1/3} \left(\int_0^1 \sin(\pi x)\,dx\right)^{2/3} \\ &\leq \left(\frac{2}{\pi}\right)^{2/3} \left(\int_0^1 u^3 \sin(\pi x)\,dx\right)^{1/3}.\end{aligned}$$

Hence

$$\int_0^1 u^3 \sin(\pi x)\,dx \geq \frac{\pi^2}{4} c^3,$$

and therefore

$$\frac{dc}{dt} \geq \pi^2 \left(-c + \frac{1}{4} c^3\right).$$

Thus, if $c(0) > 2$, Gronwall's inequality implies that

$$c(t) \geq y(t)$$

where $y(t)$ is the solution of the ODE

$$\frac{dy}{dt} = \pi^2 \left(-y + \frac{1}{4} y^3\right).$$

This solution is given explicitly by

$$y(t) = \frac{2}{\sqrt{1 - e^{2\pi^2 (t - t_*)}}}$$

This solution approaches infinity as $t \to t_*^-$ where, with $y(0) = c(0)$,

$$t_* = \frac{1}{\pi^2} \log \frac{c(0)}{\sqrt{c(0)^2 - 4}}.$$

Therefore no smooth solution of (6.33) can exist beyond $t = t_*$.

The argument used in the previous example does not prove that $c(t)$ blows up at $t = t_*$. It is conceivable that the solution loses smoothness at an earlier time — for example, because another Fourier coefficient blows up first — thereby invalidating the argument that $c(t)$ blows up. We only get a sharp result if the quantity proven to blow up is a 'controlling norm,' meaning that local smooth solutions exist so long as the controlling norm remains finite.

Example 6.12. Beale-Kato-Majda (1984) proved that solutions of the incompressible Euler equations from fluid mechanics in three-space dimensions remain smooth unless

$$\int_0^t \|\omega(s)\|_{L^\infty(\mathbb{R}^3)}\, ds \to \infty \qquad \text{as } t \to t_*^-$$

where $\omega(\cdot, t) = \operatorname{curl} \mathbf{u}(\cdot, t)$ denotes the vorticity (the curl of the fluid velocity $\mathbf{u}(\mathbf{x}, t)$). Thus, the $L^1\left(0, T; L^\infty(\mathbb{R}^3; \mathbb{R}^3)\right)$-norm of ω is a controlling norm for the three-dimensional incompressible Euler equations. It is open question whether or not this norm can blow up in finite time.