Appendix

In this appendix, we summarize some results about the integration and differentiation of Banach-space valued functions of a single variable. In a rough sense, vector-valued integrals of integrable functions have similar properties, often with similar proofs, to scalar-valued L^1-integrals. Nevertheless, the existence of different topologies (such as the weak and strong topologies) in the range space of integrals taking values in an infinite-dimensional Banach space introduces significant new issues that do not arise in the scale-valued case.

6.A. Vector-valued functions

Suppose that X is a real Banach space with norm $\|\cdot\|$ and dual space X'. Let $0 < T < \infty$, and consider functions $f : (0,T) \to X$. We will generalize some of the definitions in Section 3.A for real-valued functions of a single variable to vector-valued functions.

6.A.1. Measurability.

If $E \subset (0,T)$, let

$$\chi_E(t) = \begin{cases} 1 & \text{if } t \in E, \\ 0 & \text{if } t \notin E, \end{cases}$$

denote the characteristic function of E.

Definition 6.11. A simple function $f : (0,T) \to X$ is a function of the form

$$f = \sum_{j=1}^{N} c_j \chi_{E_j} \tag{6.33}$$

where $E_1, \dots, E_N$ are Lebesgue measurable subsets of $(0,T)$ and $c_1, \dots, c_N \in X$.

Definition 6.12. A function $f : (0,T) \to X$ is strongly measurable, or measurable for short, if there is a sequence $\{f_n : n \in \mathbb{N}\}$ of simple functions such that $f_n(t) \to f(t)$ strongly in X (*i.e.* in norm) for t *a.e.* in $(0,T)$.

Measurability is preserved under natural operations on functions.

(1) If $f : (0,T) \to X$ is measurable, then $\|f\| : (0,T) \to \mathbb{R}$ is measurable.
(2) If $f : (0,T) \to X$ is measurable and $\phi : (0,T) \to \mathbb{R}$ is measurable, then $\phi f : (0,T) \to X$ is measurable.
(3) If $\{f_n : (0,T) \to X\}$ is a sequence of measurable functions and $f_n(t) \to f(t)$ strongly in X for t pointwise *a.e.* in $(0,T)$, then $f : (0,T) \to X$ is measurable.

We will only use strongly measurable functions, but there are other definitions of measurability. For example, a function $f : (0,T) \to X$ is said to be weakly measurable if the real-valued function $\langle \omega, f\rangle : (0,T) \to \mathbb{R}$ is measurable for every $\omega \in X'$. This amounts to a 'coordinatewise' definition of measurability, in which we represent a vector-valued function by all of its possible real-valued coordinate functions. For finite-dimensional, or separable, Banach spaces these definitions coincide, but for non-separable spaces a weakly measurable function need not be strongly measurable. The relationship between weak and strong measurability is given by the following Pettis theorem (1938).

Definition 6.13. A function $f : (0, T) \to X$ taking values in a Banach space X is almost separably valued if there is a set $E \subset (0, T)$ of measure zero such that $f((0, T) \setminus E)$ is separable, meaning that it contains a dense subset.

This is equivalent to the condition that $f((0, T) \setminus E)$ lies in a closed, separable subspace of X.

Theorem 6.14. *A function $f : (0, T) \to X$ is strongly measurable if and only if it is weakly measurable and almost separably valued.*

Thus, if X is a separable Banach space, $f : (0, T) \to X$ is strongly measurable if and only $\langle \omega, f\rangle : (0, T) \to \mathbb{R}$ is measurable for every $\omega \in X'$. This theorem therefore reduces the verification of strong measurability to the verification of measurability of real-valued functions.

Definition 6.15. A function $f : [0, T] \to X$ taking values in a Banach space X is weakly continuous if $\langle \omega, f\rangle : [0, T] \to \mathbb{R}$ is continuous for every $\omega \in X'$. The space of such weakly continuous functions is denoted by $C_w([0, T]; X)$.

Since a continuous function is measurable, every almost separably valued, weakly continuous function is strongly measurable.

Example 6.16. Suppose that $\mathcal{H}$ is a non-separable Hilbert space whose dimension is equal to the cardinality of $\mathbb{R}$. Let $\{e_t : t \in (0, 1)\}$ be an orthonormal basis of $\mathcal{H}$, and define a function $f : (0, 1) \to \mathcal{H}$ by $f(t) = e_t$. Then f is weakly but not strongly measurable. If $K \subset [0, 1]$ is the standard middle thirds Cantor set and $\{\tilde{e}_t : t \in K\}$ is an orthonormal basis of $\mathcal{H}$, then $g : (0, 1) \to \mathcal{H}$ defined by $g(t) = 0$ if $t \notin K$ and $g(t) = \tilde{e}_t$ if $t \in K$ is almost separably valued since $|K| = 0$; thus, g is measurable and equivalent to the zero-function.

Example 6.17. Define $f : (0, 1) \to L^\infty(0, 1)$ by $f(t) = \chi_{(0,t)}$. Then f is not almost separably valued, since $\|f(t) - f(s)\|_{L^\infty} = 1$ for $t \neq s$, so f is not strongly measurable. On the other hand, if we define $g : (0, 1) \to L^2(0, 1)$ by $g(t) = \chi_{(0,t)}$, then g is strongly measurable. To see this, note that $L^2(0, 1)$ is separable and for every $w \in L^2(0, 1)$, which is isomorphic to $L^2(0, 1)'$, we have

$$(w, g(t))_{L^2} = \int_0^1 w(x)\chi_{(0,t)}(x)\,dx = \int_0^t w(x)\,dx.$$

Thus, $(w, g)_{L^2} : (0, 1) \to \mathbb{R}$ is absolutely continuous and therefore measurable.

6.A.2. Integration. The definition of the Lebesgue integral as a supremum of integrals of simple functions does not extend directly to vector-valued integrals because it uses the ordering properties of $\mathbb{R}$ in an essential way. One can use duality to define X-valued integrals $\int f\,dt$ in terms of the corresponding real-valued integrals $\int \langle \omega, f\rangle\,dt$ where $\omega \in X'$, but we will not consider such weak definitions of an integral here.

Instead, we define the integral of vector-valued functions by completing the space of simple functions with respect to the $L^1(0, T; X)$-norm. The resulting integral is called the Bochner integral, and its properties are similar to those of the Lebesgue integral of integrable real-valued functions. For proofs of the results stated here, see *e.g.* [**34**].

Definition 6.18. Let

$$f = \sum_{j=1}^{N} c_j \chi_{E_j}$$

be the simple function in (6.33). The integral of f is defined by

$$\int_0^T f\, dt = \sum_{j=1}^{N} c_j\, |E_j| \in X$$

where $|E_j|$ denotes the Lebesgue measure of E_j.

The value of the integral of a simple function is independent of how it is represented in terms of characteristic functions.

Definition 6.19. A strongly measurable function $f : (0,T) \to X$ is Bochner integrable, or integrable for short, if there is a sequence of simple functions such that $f_n(t) \to f(t)$ pointwise *a.e.* in $(0,T)$ and

$$\lim_{n\to\infty} \int_0^T \|f - f_n\|\, dt = 0.$$

The integral of f is defined by

$$\int_0^T f\, dt = \lim_{n\to\infty} \int_0^T f_n\, dt,$$

where the limit exists strongly in X.

The value of the Bochner integral of f is independent of the sequence $\{f_n\}$ of approximating simple functions, and

$$\left\| \int_0^T f\, dt \right\| \le \int_0^T \|f\|\, dt.$$

Moreover, if $A : X \to Y$ is a bounded linear operator between Banach spaces X, Y and $f : (0,T) \to X$ is integrable, then $Af : (0,T) \to Y$ is integrable and

$$A\left(\int_0^T f\, dt\right) = \int_0^T Af\, dt. \tag{6.34}$$

More generally, this equality holds whenever $A : \mathcal{D}(A) \subset X \to Y$ is a closed linear operator and $f : (0,T) \to \mathcal{D}(A)$, in which case $\int_0^T f\, dt \in \mathcal{D}(A)$.

Example 6.20. If $f : (0,T) \to X$ is integrable and $\omega \in X'$, then $\langle \omega, f\rangle : (0,T) \to \mathbb{R}$ is integrable and

$$\left\langle \omega, \int_0^T f\, dt \right\rangle = \int_0^T \langle \omega, f\rangle\, dt.$$

Example 6.21. If $J : X \hookrightarrow Y$ is a continuous embedding of a Banach space X into a Banach space Y, and $u : (0,T) \to X$, then

$$J\left(\int_0^T u\, dt\right) = \int_0^T Ju\, dt.$$

Thus, the X and Y valued integrals agree and we can identify them.

The following result, due to Bochner (1933), characterizes integrable functions as ones with integrable norm.

Theorem 6.22. *A function $f : (0,T) \to X$ is Bochner integrable if and only if it is strongly measurable and*

$$\int_0^T \|f\| \, dt < \infty.$$

Thus, in order to verify that a measurable function f is Bochner integrable one only has to check that the real valued function $\|f\| : (0,T) \to \mathbb{R}$, which is necessarily measurable, is integrable.

Example 6.23. The functions $f : (0,1) \to \mathcal{H}$ in Example (6.16) and $f : (0,1) \to L^\infty(0,1)$ in Example (6.17) are not Bochner integrable since they are not strongly measurable. The function $g : (0,1) \to \mathcal{H}$ in Example (6.16) is Bochner integrable, and its integral is equal to zero. The function $g : (0,1) \to L^2(0,1)$ in Example (6.17) is Bochner integrable since it is measurable and $\|g(t)\|_{L^2} = t^{1/2}$ is integrable on $(0,1)$. We leave it as an exercise to compute its integral.

The dominated convergence theorem holds for Bochner integrals. The proof is the same as for the scalar-valued case, and we omit it.

Theorem 6.24. *Suppose that $f_n : (0,T) \to X$ is Bochner integrable for each $n \in \mathbb{N}$,*

$$f_n(t) \to f(t) \qquad \text{as } n \to \infty \text{ strongly in } X \text{ for } t \text{ a.e. in } (0,T),$$

and there is an integrable function $g : (0,T) \to \mathbb{R}$ such that

$$\|f_n(t)\| \leq g(t) \qquad \text{for } t \text{ a.e. in } (0,T) \text{ and every } n \in \mathbb{N}.$$

Then $f : (0,T) \to X$ is Bochner integrable and

$$\int_0^T f_n \, dt \to \int_0^T f \, dt, \qquad \int_0^T \|f_n - f\| \, dt \to 0 \qquad \text{as } n \to \infty.$$

The definition and properties of L^p-spaces of X-valued functions are analogous to the case of real-valued functions.

Definition 6.25. For $1 \leq p < \infty$ the space $L^p(0,T;X)$ consists of all strongly measurable functions $f : (0,T) \to X$ such that

$$\int_0^T \|f\|^p \, dt < \infty$$

equipped with the norm

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f\|^p \, dt \right)^{1/p}.$$

The space $L^\infty(0,T;X)$ consists of all strongly measurable functions $f : (0,T) \to X$ such that

$$\|f\|_{L^\infty(0,T;X)} = \sup_{t \in (0,T)} \|f(t)\| < \infty,$$

where sup denotes the essential supremum.

As usual, we regard functions that are equal pointwise *a.e.* as equivalent, and identify a function that is equivalent to a continuous function with its continuous representative.

Theorem 6.26. *If X is a Banach space and $1 \le p \le \infty$, then $L^p(0,T;X)$ is a Banach space.*

Simple functions of the form

$$f(t) = \sum_{i=1}^{n} c_i \chi_{E_i}(t),$$

where $c_i \in X$ and E_i is a measurable subset of $(0,T)$, are dense in $L^p(0,T;X)$. By mollifying these functions with respect to t, we get the following density result.

Proposition 6.27. *If X is a Banach space and $1 \le p < \infty$, then the collection of functions of the form*

$$f(t) = \sum_{i=1}^{n} c_i \phi_i(t) \qquad \text{where } \phi_i \in C_c^\infty(0,T) \text{ and } c_i \in X$$

is dense in $L^p(0,T;X)$.

The characterization of the dual space of a vector-valued L^p-space is analogous to the scalar-valued case, after we take account of duality in the range space X.

Theorem 6.28. *Suppose that $1 \le p < \infty$ and X is a reflexive Banach space with dual space X'. Then the dual of $L^p(0,T;X)$ is isomorphic to $L^{p'}(0,T;X')$ where*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The action of $f \in L^{p'}(0,T;X')$ on $u \in L^p(0,T;X)$ is given by

$$\langle\langle f, u \rangle\rangle = \int_0^T \langle f(t), u(t) \rangle \, dt,$$

where the double angles denote the $L^p(X)$-$L^{p'}(X')$ duality pairing and the single brackets denote the X-X' duality pairing.

The proof is more complicated than in the scalar case, and we need to impose some condition on X. Reflexivity is sufficient (as is the condition that X' is separable).

6.A.3. Differentiability. The definition of continuity and pointwise differentiability of vector-valued functions are the same as in the scalar case. A function $f : (0,T) \to X$ is strongly continuous at $t \in (0,T)$ if $f(s) \to f(t)$ strongly in X as $s \to t$, and f is strongly continuous in $(0,T)$ if it is strongly continuous at every point of $(0,T)$. A function f is strongly differentiable at $t \in (0,T)$, with strong pointwise derivative $f_t(t)$, if

$$f_t(t) = \lim_{h \to 0} \left[\frac{f(t+h) - f(t)}{h} \right]$$

where the limit exists strongly in X, and f is continuously differentiable in $(0,T)$ if its pointwise derivative exists for every $t \in (0,T)$ and $f_t : (0,T) \to X$ is a strongly continuously function.

The assumption of continuous differentiability is often too strong to be useful, so we need a weaker notion of the differentiability of a vector-valued function. As for real-valued functions, such as the step function or the Cantor function, the requirement that the strong pointwise derivative exists *a.e.* in $(0,T)$ does *not* lead to an effective theory. Instead we use the notion of a distributional or weak derivative, which is a natural generalization of the definition for real-valued functions.

Let $L^1_{\mathrm{loc}}(0,T;X)$ denote the space of measurable functions $f:(0,T)\to X$ that are integrable on every compactly supported interval $(a,b)\Subset(0,T)$. Also, as usual, let $C_c^\infty(0,T)$ denote the space of smooth, real-valued functions $\phi:(0,T)\to\mathbb{R}$ with compact support $\operatorname{spt}\phi\Subset(0,T)$.

Definition 6.29. A function $f\in L^1_{\mathrm{loc}}(0,T;X)$ is weakly differentiable with weak derivative $f_t=g\in L^1_{\mathrm{loc}}(0,T;X)$ if

$$\int_0^T \phi' f\,dt = -\int_0^T \phi g\,dt \qquad \text{for every } \phi\in C_c^\infty(0,T). \tag{6.35}$$

The integrals in (6.35) are understood to be Bochner integrals. In the commonly occurring case where $J:X\hookrightarrow Y$ is a continuous embedding, $f\in L^1_{\mathrm{loc}}(0,T;X)$, and $(Jf)_t\in L^1_{\mathrm{loc}}(0,T;Y)$, we have from Example 6.21 that

$$J\left(\int_0^T \phi' f\,dt\right) = \int_0^T \phi' Jf\,dt = -\int_0^T \phi (Jf)_t\,dt.$$

Thus, we can identify f with Jf and use (6.35) to define the Y-valued derivative of an X-valued function.

If $f:(0,T)\to\mathbb{R}$ is a scalar-valued, integrable function, then the Lebesgue differentiation theorem, Theorem 1.21, implies that the limit

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s)\,ds$$

exists and is equal to $f(t)$ for t pointwise *a.e.* in $(0,T)$. The same result is true for vector-valued integrals.

Theorem 6.30. *Suppose that X is a Banach space and $f\in L^1(0,T;X)$, then*

$$f(t)=\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s)\,ds$$

for t pointwise a.e. in $(0,T)$.

Proof. Since f is almost separably valued, we may assume that X is separable. Let $\{c_n\in X:n\in\mathbb{N}\}$ be a dense subset of X, then by the Lebesgue differentiation theorem for real-valued functions

$$\|f(t)-c_n\| = \lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\|f(s)-c_n\|\,ds$$

for every $n\in\mathbb{N}$ and t pointwise *a.e.* in $(0,T)$. Thus, for all such $t\in(0,T)$ and every $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\limsup_{h\to 0}\frac{1}{h}\int_t^{t+h}\|f(s)-f(t)\|\,ds\\
&\qquad\qquad \le \limsup_{h\to 0}\frac{1}{h}\int_t^{t+h}\left(\|f(s)-c_n\|+\|f(t)-c_n\|\right)ds\\
&\qquad\qquad \le 2\,\|f(t)-c_n\|\,.
\end{aligned}$$

Since this holds for every c_n, it follows that

$$\limsup_{h\to 0} \frac{1}{h}\int_t^{t+h} \|f(s) - f(t)\|\, ds = 0.$$

Therefore

$$\limsup_{h\to 0} \left\| \frac{1}{h}\int_t^{t+h} f(s)\, ds - f(t)\right\| \leq \limsup_{h\to 0} \frac{1}{h}\int_t^{t+h} \|f(s) - f(t)\|\, ds = 0,$$

which proves the result. □

The following corollary corresponds to the statement that a regular distribution determines the values of its associated locally integrable function pointwise almost everywhere.

Corollary 6.31. *Suppose that $f : (0,T) \to X$ is locally integrable and*

$$\int_0^T \phi f\, dt = 0 \qquad \textit{for every } \phi \in C_c^\infty(0,T).$$

Then $f = 0$ pointwise a.e. on $(0,T)$.

Proof. Choose a sequence of test functions $0 \leq \phi_n \leq 1$ whose supports are contained inside a fixed compact subset of $(0,T)$ such that $\phi_n \to \chi_{(t,t+h)}$ pointwise, where $\chi_{(t,t+h)}$ is the characteristic function of the interval $(t, t+h) \subset (0,T)$. If $f \in L^1_{\text{loc}}(0,T;X)$, then by the dominated convergence theorem

$$\int_t^{t+h} f(s)\, ds = \lim_{n\to\infty} \int_0^T \phi_n(s) f(s)\, ds.$$

Thus, if $\int_0^T \phi f\, ds = 0$ for every $\phi \in C_c^\infty(0,T)$, then

$$\int_t^{t+h} f(s)\, ds = 0$$

for every $(t, t+h) \subset (0,T)$. It then follows from the Lebesgue differentiation theorem, Theorem 6.30, that $f = 0$ pointwise *a.e.* in $(0,T)$. □

We also have a vector-valued analog of Proposition 3.6 that the only functions with zero weak derivative are the constant functions. The proof is similar.

Proposition 6.32. *Suppose that $f : (0,T) \to X$ is weakly differentiable and $f' = 0$. Then f is equivalent to a constant function.*

Proof. The condition that the weak derivative f' is zero means that

$$\int_0^T f\phi'\, dt = 0 \qquad \text{for all } \phi \in C_c^\infty(0,T). \tag{6.36}$$

Choose a fixed test function $\eta \in C_c^\infty(0,T)$ whose integral is equal to one, and represent an arbitrary test function $\phi \in C_c^\infty(0,T)$ as

$$\phi = A\eta + \psi'$$

where $A \in \mathbb{R}$ and $\psi \in C_c^\infty(0,T)$ are given by

$$A = \int_0^T \phi\, dt, \qquad \psi(t) = \int_0^t [\phi(s) - A\eta(s)]\, ds.$$

If

$$c = \int_0^T \eta f \, dt \in X,$$

then (6.36) implies that

$$\int_0^T (f - c)\, \phi \, dt = 0 \qquad \text{for all } \phi \in C_c^\infty(0,T), \tag{6.37}$$

and Corollary 6.31 implies that $f = c$ pointwise *a.e.* on $(0,T)$. □

It also follows that a function is weakly differentiable if and only if it is the integral of an integrable function.

Theorem 6.33. *Suppose that X is a Banach space and $f \in L^1(0,T;X)$. Then u is weakly differentiable with integrable derivative $f_t = g \in L^1(0,T;X)$ if and only if*

$$f(t) = c_0 + \int_0^t g(s)\, ds \tag{6.38}$$

pointwise a.e. in $(0,T)$. In that case, u is differentiable pointwise a.e. and its pointwise derivative coincides with its weak derivative.

Proof. If f is given by (6.38), then

$$\frac{f(t+h) - f(t)}{h} = \frac{1}{h}\int_t^{t+h} g(s)\, ds,$$

and the Lebesgue differentiation theorem, Theorem 6.30, implies that the strong derivative of f exists pointwise *a.e.* and is equal to g.

We also have that

$$\left\| \frac{f(t+h) - f(t)}{h} \right\| \le \frac{1}{h}\int_t^{t+h} \|g(s)\|\, ds.$$

Extending f by zero to a function $f : \mathbb{R} \to X$, and using Fubini's theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}} \left\| \frac{f(t+h) - f(t)}{h} \right\| dt &\le \frac{1}{h}\int_{\mathbb{R}} \left(\int_t^{t+h} \|g(s)\|\, ds \right) dt \\
&\le \frac{1}{h}\int_{\mathbb{R}} \left(\int_0^{h} \|g(s+t)\|\, ds \right) dt \\
&\le \frac{1}{h}\int_0^{h} \left(\int_{\mathbb{R}} \|g(s+t)\|\, dt \right) ds \\
&\le \int_{\mathbb{R}} \|g(t)\|\, dt.
\end{aligned}$$

If $\phi \in C_c^\infty(0,T)$, this estimate justifies the use of the dominated convergence theorem and the previous result on the pointwise *a.e.* convergence of f_t to get

$$\begin{aligned}
\int_0^T \phi'(t) f(t)\, dt &= \lim_{h\to 0} \int_0^T \left[\frac{\phi(t+h) - \phi(t)}{h} \right] f(t)\, dt \\
&= -\lim_{h\to 0} \int_0^T \phi(t) \left[\frac{f(t) - f(t-h)}{h} \right] dt \\
&= -\int_0^T \phi(t) g(t)\, dt,
\end{aligned}$$

which shows that g is the weak derivative of f.

Conversely, if $f_t = g \in L^1(0,T)$ in the sense of weak derivatives, let

$$\tilde{f}(t) = \int_0^t g(s)\, ds.$$

Then the previous argument implies that $\tilde{f}_t = g$, so the weak derivative $(f - \tilde{f})_t$ is zero. Proposition 6.32 then implies that $f - \tilde{f}$ is constant pointwise *a.e.*, which gives (6.38). □

We can also characterize the weak derivative of a vector-valued function in terms of weak derivatives of the real-valued functions obtained by duality.

Proposition 6.34. *Let X be a Banach space with dual X'. If $f, g \in L^1(0,T;X)$, then f is weakly differentiable with $f_t = g$ if and only if for every $\omega \in X'$*

$$\frac{d}{dt}\langle \omega, f\rangle = \langle \omega, g\rangle \qquad \text{as a real-valued weak derivative in } (0,T). \tag{6.39}$$

Proof. If $f_t = g$, then

$$\int_0^T \phi' f\, dt = -\int_0^T \phi g\, dt \qquad \text{for all } \phi \in C_c^\infty(0,T).$$

Acting on this equation by $\omega \in X'$ and using the continuity of the integral, we get

$$\int_0^T \phi'\langle \omega, f\rangle\, dt = -\int_0^T \phi\langle \omega, g\rangle\, dt \qquad \text{for all } \phi \in C_c^\infty(0,T)$$

which is (6.39). Conversely, if (6.39) holds, then

$$\left\langle \omega, \int_0^T (\phi' f + \phi g)\ dt \right\rangle = 0 \qquad \text{for all } \omega \in X',$$

which implies that

$$\int_0^T (\phi' f + \phi g)\ dt = 0.$$

Therefore f is weakly differentiable with $f_t = g$. □

A consequence of these results is that any of the natural ways of defining what one means for an abstract evolution equation to hold in a weak sense leads to the same notion of a solution. To be more explicit, suppose that $X \hookrightarrow Y$ are Banach spaces with X continuously and densely embedded in Y and $F : X \times (0,T) \to Y$. Then a function $u \in L^1(0,T;X)$ is a weak solution of the equation

$$u_t = F(u,t)$$

if it has a weak derivative $u_t \in L^1(0,T;Y)$ and $u_t = F(u,t)$ for t pointwise *a.e.* in $(0,T)$. Equivalent ways of stating this property are that

$$u(t) = u_0 + \int_0^t F(u(s),s)\ ds \qquad \text{for } t \text{ pointwise } a.e. \text{ in } (0,T);$$

or that

$$\frac{d}{dt}\langle \omega, u(t)\rangle = \langle \omega, F(u(t),t)\rangle \qquad \text{for every } \omega \in Y'$$

in the sense of real-valued weak derivatives. Moreover, by approximating arbitrary smooth functions $w : (0,T) \to Y'$ by linear combinations of functions of the form $w(t) = \phi(t)\omega$, we see that this is equivalent to the statement that

$$-\int_0^T \langle w_t(t), u(t)\rangle \, dt = \int_0^T \langle w(t), F(u(t), t)\rangle \, dt \qquad \text{for every } w \in C_c^\infty(0,T;Y').$$

We define Sobolev spaces of vector-valued functions in the same way as for scalar-valued functions, and they have similar properties.

Definition 6.35. Suppose that X is a Banach space, $k \in \mathbb{N}$, and $1 \le p \le \infty$. The Banach space $W^{k,p}(0,T;X)$ consists of all (equivalence classes of) measurable functions $u : (0,T) \to X$ whose weak derivatives of order $0 \le j \le k$ belong to $L^p(0,T;X)$. If $1 \le p < \infty$, then the $W^{k,p}$-norm is defined by

$$\|u\|_{W^{k,p}(0,T;X)} = \left(\sum_{j=1}^{k} \left\|\partial_t^j u\right\|_X^p \, dt\right)^{1/p};$$

if $p = \infty$, then

$$\|u\|_{W^{k,p}(0,T;X)} = \sup_{1\le j\le k} \left\|\partial_t^j u\right\|_X .$$

If $p = 2$, and $X = \mathcal{H}$ is a Hilbert space, then $W^{k,2}(0,T;\mathcal{H}) = H^k(0,T;\mathcal{H})$ is the Hilbert space with inner product

$$(u,v)_{H^k(0,T;\mathcal{H})} = \int_0^T (u(t), v(t))_{\mathcal{H}} \, dt.$$

The Sobolev embedding theorem for scalar-valued functions of a single variable carries over to the vector-valued case.

Theorem 6.36. *If $1 \le p \le \infty$ and $u \in W^{1,p}(0,T;X)$, then $u \in C([0,T];X)$. Moreover, there exists a constant $C = C(p,T)$ such that*

$$\|u\|_{L^\infty(0,T;X)} \le C \|u\|_{W^{1,p}(0,T;X)} .$$

PROOF. From Theorem 6.33, we have

$$\|u(t) - u(s)\| \le \int_s^t \|u_t(r)\| \, dr.$$

Since $\|u_t\| \in L^1(0,T)$, its integral is absolutely continuous, so u is uniformly continuous on $(0,T)$ and extends to a continuous function on $[0,T]$.

If $h : (0,T) \to \mathbb{R}$ is defined by $h = \|u\|$, then

$$|h(t) - h(s)| \le \|u(t) - u(s)\| \le \int_s^t \|u_t(r)\| \, dr.$$

It follows that h is absolutely continuous and $|h_t| \le \|u_t\|$ pointwise a.e. on $(0,T)$. Therefore, by the Sobolev embedding theorem for real valued functions,

$$\|u\|_{L^\infty(0,T;X)} = \|h\|_{L^\infty(0,T)} \le C \|h\|_{W^{1,p}(0,T)} \le C \|u\|_{W^{1,p}(0,T;X)} .$$

□

6.A.4. The Radon-Nikodym property. Although we do not use this discussion elsewhere, it is interesting to consider the relationship between weak differentiability and absolute continuity in the vector-valued case.

The definition of absolute continuity of vector-valued functions is a natural generalization of the real-valued definition. We say that $f : [0,T] \to X$ is absolutely continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{n=1}^{N} \|f(t_n) - f(t_{n-1})\| < \epsilon$$

for every collection $\{[t_0, t_1], [t_2, t_1], \dots, [t_{N-1}, t_N]\}$ of non-overlapping subintervals of $[0,T]$ such that

$$\sum_{n=1}^{N} |t_n - t_{n-1}| < \delta.$$

Similarly, $f : [0,T] \to X$ is Lipschitz continuous on $[0,T]$ if there exists a constant $M \geq 0$ such that

$$\|f(s) - f(t)\| \leq M|s-t| \qquad \text{for all } s, t \in [0,T].$$

It follows immediately that a Lipschitz continuous function is absolutely continuous (with $\delta = \epsilon/M$).

A real-valued function is weakly differentiable with integrable derivative if and only if it is absolutely continuous *c.f.* Theorem 3.60. This is one of the few properties of real-valued integrals that does not carry over to Bochner integrals in arbitrary Banach spaces. It follows from the integral representation in Theorem 6.33 that every weakly differentiable function with integrable derivative is absolutely continuous, but it can happen that an absolutely continuous vector-valued function is not weakly differentiable.

Example 6.37. Define $f : (0,1) \to L^\infty(0,1)$ by

$$f(t) = t\chi_{[0,t]}.$$

Then f is Lipschitz continuous, and therefore absolutely continuous. Nevertheless, the derivative $f'(t)$ does not exist for any $t \in (0,1)$ since the limit as $h \to 0$ of the difference quotient

$$\frac{f(t+h) - f(t)}{h}$$

does not converge in $L^\infty(0,1)$, so by Theorem 6.33 f is not weakly differentiable.

A Banach space for which every absolutely continuous function has an integrable weak derivative is said to have the Radon-Nikodym property. Any reflexive Banach space has this property but, as the previous example shows, the space $L^\infty(0,1)$ does not. One can use the Radon-Nikodym property to study the geometric structure of Banach spaces, but this question is not relevant for our purposes. Most of the spaces we use are reflexive, and even if they are not, we do not need an explicit characterization of the weakly differentiable functions.

6.B. Hilbert triples

Hilbert triples provide a useful framework for the study of weak and variational solutions of PDEs. We consider real Hilbert spaces for simplicity. For complex Hilbert spaces, one has to replace duals by antiduals, as appropriate.

Definition 6.38. A Hilbert triple consists of three separable Hilbert spaces

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$$

such that $\mathcal{V}$ is densely embedded in $\mathcal{H}$, $\mathcal{H}$ is densely embedded in $\mathcal{V}'$, and

$$\langle f, v\rangle = (f, v)_{\mathcal{H}} \qquad \text{for every } f \in \mathcal{H} \text{ and } v \in \mathcal{V}.$$

Hilbert triples are also referred to as Gelfand triples, variational triples, or rigged Hilbert spaces. In this definition, $\langle \cdot, \cdot\rangle : \mathcal{V}' \times \mathcal{V} \to \mathbb{R}$ denotes the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$, and $(\cdot, \cdot)_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ denotes the inner product on $\mathcal{H}$. Thus, we identify: (a) the space $\mathcal{V}$ with a dense subspace of $\mathcal{H}$ through the embedding; (b) the dual of the 'pivot' space $\mathcal{H}$ with itself through its own inner product, as usual for a Hilbert space; (c) the space $\mathcal{H}$ with a subspace of the dual space $\mathcal{V}'$, where $\mathcal{H}$ acts on $\mathcal{V}$ through the $\mathcal{H}$-inner product, *not* the $\mathcal{V}$-inner product.

In the elliptic and parabolic problems considered above involving a uniformly elliptic, second-order operator, we have

$$\mathcal{V} = H_0^1(\Omega), \qquad \mathcal{H} = L^2(\Omega), \qquad \mathcal{V}' = H^{-1}(\Omega),$$
$$(f, g)_{\mathcal{H}} = \int_\Omega fg\,dx, \quad (f, g)_{\mathcal{V}} = \int_\Omega Df \cdot Dg\,dx,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set. Nothing will be lost by thinking about this case. The embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is inclusion. The embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is defined by the identification of an L^2-function with its corresponding regular distribution, and the action of $f \in L^2(\Omega)$ on a test function $v \in H_0^1(\Omega)$ is given by

$$\langle f, v\rangle = \int_\Omega fv\,dx.$$

The isomorphism between $\mathcal{V}$ and its dual space $\mathcal{V}'$ is then given by

$$-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega).$$

Thus, a Hilbert triple allows us to represent a 'concrete' operator, such as $-\Delta$, as an isomorphism between a Hilbert space and its dual.

As suggested by this example, in studying evolution equations such as the heat equation $u_t = \Delta u$, we are interested in functions u that take values in $\mathcal{V}$ whose weak time-derivatives u_t takes values in $\mathcal{V}'$. The basic facts about such functions are given in the next theorem, which states roughly that the natural identities for time derivatives hold provided that the duality pairings they involve make sense.

Theorem 6.39. *Let $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$ be a Hilbert triple. If $u \in L^2(0, T; \mathcal{V})$ and $u_t \in L^2(0, T; \mathcal{V}')$, then $u \in C([0, T]; \mathcal{H})$. Moreover:*

(1) *for any $v \in \mathcal{V}$, the real-valued function $t \mapsto (u(t), v)_{\mathcal{H}}$ is weakly differentiable in $(0, T)$ and*

$$\frac{d}{dt}(u(t), v)_{\mathcal{H}} = \langle u_t(t), v\rangle; \tag{6.40}$$

(2) *the real-valued function $t \mapsto \|u(t)\|_{\mathcal{H}}^2$ is weakly differentiable in $(0, T)$ and*

$$\frac{d}{dt}\|u\|_{\mathcal{H}}^2 = 2\langle u_t, u\rangle; \tag{6.41}$$

(3) *there is a constant $C = C(T)$ such that*

$$\|u\|_{L^\infty(0,T;\mathcal{H})} \le C\left(\|u\|_{L^2(0,T;\mathcal{V})} + \|u_t\|_{L^2(0,T;\mathcal{V}')}\right). \tag{6.42}$$

Proof. We extend u to a map $u : (-\infty, \infty) \to \mathcal{V}$ by defining $u(t) = 0$ for $t \notin (0,T)$ and mollify the extension with the standard mollifier $\eta^\epsilon : \mathbb{R} \to \mathbb{R}$ to obtain a smooth approximation

$$u^\epsilon = \eta^\epsilon * u \in C_c^\infty(\mathbb{R}; \mathcal{V}), \qquad u^\epsilon(t) = \int_{-\infty}^{\infty} \eta^\epsilon(t-s) u(s)\, ds.$$

The same results that apply to mollifiers of real-valued functions apply to these vector-valued functions. As $\epsilon \to 0^+$, we have: $u^\epsilon \to u$ in $L^2(0,T;\mathcal{V})$, $u_t^\epsilon = \eta^\epsilon * u_t \to u_t$ in $L^2(0,T;\mathcal{V}')$, and $u^\epsilon(t) \to u(t)$ in $\mathcal{V}$ for t pointwise *a.e.* in $(0,T)$. Moreover, as a consequence of the boundedness of the extension operator and the fact that mollification does not increase the norm of a function, there exists a constant $0 < C < 1$ such that for all $0 < \epsilon \leq 1$, say,

$$C \left\| u^\epsilon \right\|_{L^2(\mathbb{R};\mathcal{V})} \leq \left\| u \right\|_{L^2(0,T;\mathcal{V})} \leq \left\| u^\epsilon \right\|_{L^2(\mathbb{R};\mathcal{V})}. \tag{6.43}$$

Since u^ϵ is a smooth $\mathcal{V}$-valued function and $\mathcal{V} \hookrightarrow \mathcal{H}$, we have

$$(u^\epsilon(t), u^\epsilon(t))_{\mathcal{H}} = \int_{-\infty}^{t} \frac{d}{ds} (u^\epsilon(s), u^\epsilon(s))_{\mathcal{H}}\, ds = 2 \int_{-\infty}^{t} (u_s^\epsilon(s), u^\epsilon(s))_{\mathcal{H}}\, ds. \tag{6.44}$$

Using the analogous formula for $u^\epsilon - u^\delta$, the duality estimate and the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
\left\| u^\epsilon(t) - u^\delta(t) \right\|_{\mathcal{H}}^2 &\leq 2 \int_{-\infty}^{\infty} \left\| u_s^\epsilon(s) - u_s^\delta(s) \right\|_{\mathcal{V}'} \left\| u^\epsilon(s) - u^\delta(s) \right\|_{\mathcal{V}} ds \\
&\leq 2 \left\| u_t^\epsilon - u_t^\delta \right\|_{L^2(\mathbb{R};\mathcal{V}')} \left\| u^\epsilon - u^\delta \right\|_{L^2(\mathbb{R};\mathcal{V})}.
\end{aligned}$$

Since $\{u^\epsilon\}$ is Cauchy in $L^2(\mathbb{R};\mathcal{V})$ and $\{u_t^\epsilon\}$ is Cauchy in $L^2(\mathbb{R};\mathcal{V}')$, it follows that $\{u^\epsilon\}$ is Cauchy in $C_c(\mathbb{R};\mathcal{H})$, and therefore converges uniformly on $[0,T]$ to a function $v \in C([0,T];\mathcal{H})$. Since u^ϵ converges pointwise *a.e.* to u, it follows that u is equivalent to v, so $u \in C([0,T];\mathcal{H})$ after being redefined, if necessary, on a set of measure zero.

Taking the limit of (6.44) as $\epsilon \to 0^+$, we find that for $t \in [0,T]$

$$\left\| u(t) \right\|_{\mathcal{H}}^2 = \left\| u(0) \right\|_{\mathcal{H}}^2 + 2 \int_0^t \langle u_s(s), u(s) \rangle\, ds,$$

which implies that $\|u\|_{\mathcal{H}}^2 : [0,T] \to \mathbb{R}$ is absolutely continuous and (6.41) holds. Moreover, (6.42) follows from (6.43), (6.44), and the Cauchy-Schwartz inequality.

Finally, if $\phi \in C_c^\infty(0,T)$ is a test function $\phi : (0,T) \to \mathbb{R}$ and $v \in \mathcal{V}$, then $\phi v \in C_c^\infty(0,T;\mathcal{V})$. Therefore, since $u_t^\epsilon \to u_t$ in $L^2(0,T;\mathcal{V}')$,

$$\int_0^T \langle u_t^\epsilon, \phi v \rangle\, dt \to \int_0^T \langle u_t, \phi v \rangle\, dt.$$

Also, since u^ϵ is a smooth $\mathcal{V}$-valued function,

$$\int_0^T \langle u_t^\epsilon, \phi v \rangle\, dt = -\int_0^T \phi' \langle u^\epsilon, v \rangle\, dt \to -\int_0^T \phi' \langle u, v \rangle\, dt$$

We conclude that for every $\phi \in C_c^\infty(0,T)$ and $v \in \mathcal{V}$

$$\int_0^T \phi \langle u_t, v \rangle\, dt = -\int_0^T \phi_t \langle u, v \rangle\, dt$$

which is the weak form of (6.40). □

We further have the following integration by parts formula

Theorem 6.40. *Suppose that* $u, v \in L^2(0,T;\mathcal{V})$ *and* $u_t, v_t \in L^2(0,T;\mathcal{V}')$. *Then*

$$\int_0^T \langle u_t, v\rangle \, dt = (u(T), v(T))_{\mathcal{H}} - (u(0), v(0))_{\mathcal{H}} - \int_0^T \langle u, v_t\rangle \, dt.$$

Proof. This result holds for smooth functions $u, v \in C^\infty([0,T];\mathcal{V})$. Therefore by density and Theorem 6.39 it holds for all functions $u, v \in L^2(0,T;\mathcal{V})$ with $u_t, v_t \in L^2(0,T;\mathcal{V}')$. □