CHAPTER 7

Hyperbolic Equations

Hyperbolic PDEs arise in physical applications as models of waves, such as acoustic, elastic, electromagnetic, or gravitational waves. The qualitative properties of hyperbolic PDEs differ sharply from those of parabolic PDEs. For example, they have finite domains of influence and dependence, and singularities in solutions propagate without being smoothed.

7.1. The wave equation

The prototypical example of a hyperbolic PDE is the wave equation

$$u_{tt} = \Delta u. \tag{7.1}$$

To begin with, consider the one-dimensional wave equation on $\mathbb{R}$,

$$u_{tt} = u_{xx}.$$

The general solution is the d'Alembert solution

$$u(x,t) = f(x-t) + g(x+t)$$

where f, g are arbitrary functions, as one may verify directly. This solution describes a superposition of two traveling waves with arbitrary profiles, one propagating with speed one to the right, the other with speed one to the left.

Let us compare this solution with the general solution of the one-dimensional heat equation

$$u_t = u_{xx},$$

which is given for $t > 0$ by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/4t} f(y)\, dy.$$

Some of the qualitative properties of the wave equation that differ from those of the heat equation, which are evident from these solutions, are:

(1) the wave equation has finite propagation speed and domains of influence;
(2) the wave equation is reversible in time;
(3) solutions of the wave equation do not become smoother in time;
(4) the wave equation do not satisfy a maximum principle.

A suitable IBVP for the wave equation with Dirichlet BCs on a bounded open set $\Omega \subset \mathbb{R}^n$ for $u : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} &u_{tt} = \Delta u && \text{for } x \in \Omega \text{ and } t \in \mathbb{R}, \\ &u(x,t) = 0 && \text{for } x \in \partial\Omega \text{ and } t \in \mathbb{R}, \\ &u(x,0) = g(x), \quad u_t(x,0) = h(x) && \text{for } x \in \Omega. \end{aligned} \tag{7.2}$$

We require two initial conditions since the wave equation is second-order in time. For example, in two space dimensions, this IBVP would describe the small vibrations of an elastic membrane, with displacement $z = u(x, y, t)$, such as a drum. The membrane is fixed at its edge $\partial\Omega$, and has initial displacement g and initial velocity h. We could also add a nonhomogeneous term to the PDE, which would describe an external force, but we omit it for simplicity.

7.1.1. Energy estimate. To obtain the basic energy estimate for the wave equation, we multiple (7.1) by u_t and write

$$\begin{aligned} u_t u_{tt} &= \left(\frac{1}{2}u_t\right)_t, \\ u_t \Delta u &= \operatorname{div}(u_t Du) - Du \cdot Du_t = \operatorname{div}(u_t Du) - \left(\frac{1}{2}|Du|^2\right)_t \end{aligned}$$

to get

$$\left(\frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2\right) - \operatorname{div}(u_t Du) = 0. \tag{7.3}$$

This is the differential form of conservation of energy. The quantity $\frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2$ is the energy density (kinetic plus potential energy) and $-u_t Du$ is the energy flux.

If u is a solution of (7.2), then integration of (7.3) over Ω, use of the divergence theorem, and the BC $u = 0$ on $\partial\Omega$ (which implies that $u_t = 0$) gives

$$\frac{dE}{dt} = 0$$

where $E(t)$ is the total energy

$$E(t) = \int_\Omega \left(\frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2\right) dx.$$

Thus, the total energy remains constant. This result provides an L^2-energy estimate for solutions of the wave equation.

We will use this estimate to construct weak solutions of a general wave equation by a Galerkin method. Despite the qualitative difference in the properties of parabolic and hyperbolic PDEs, the proof is similar to the proof in Chapter 6 for the existence of weak solutions of parabolic PDEs. Some of the details are, however, more delicate; the lack of smoothing of hyperbolic PDEs is reflected analytically by weaker estimates for their solutions. For additional discussion see [**27**].

7.2. Definition of weak solutions

We consider a uniformly elliptic, second-order operator of the form (6.5). For simplicity, we assume that $b^i = 0$. In that case,

$$Lu = -\sum_{i,j=1}^n \partial_i \left(a^{ij}(x,t)\partial_j u\right) + c(x,t)u, \tag{7.4}$$

and L is formally self-adjoint. The first-order spatial derivative terms would be straightforward to include, at the expense of complicating the energy estimates somewhat. We could also include appropriate first-order time derivatives in the equation proportional to u_t.

Generalizing (7.2), we consider the following IBVP for a second-order hyperbolic PDE

$$
\begin{aligned}
&u_{tt} + Lu = f && \text{in } \Omega \times (0,T),\\
&u = 0 && \text{on } \partial\Omega \times (0,T),\\
&u = g, \quad u_t = h && \text{on } t = 0.
\end{aligned}
\tag{7.5}
$$

To formulate a definition of a weak solution of (7.5), let $a(u,v;t) = (Lu,v)_{L^2}$ be the bilinear form associated with L in (7.4),

$$
a(u,v;t) = \sum_{i,j=1}^{n} \int_\Omega a^{ij}(x,t)\partial_i u(x)\partial_j u(x)\,dx + \int_\Omega c(x,t)u(x)v(x)\,dx. \tag{7.6}
$$

We make the following assumptions.

Assumption 7.1. *The set $\Omega \subset \mathbb{R}^n$ is bounded and open, $T > 0$, and:*

1. *the coefficients of a in (7.6) satisfy*

$$
a^{ij}, c \in L^\infty(\Omega \times (0,T)), \qquad a^{ij}_t, c_t \in L^\infty(\Omega \times (0,T));
$$

2. *$a^{ij} = a^{ji}$ for $1 \le i,j \le n$ and the uniform ellipticity condition (6.6) holds for some constant $\theta > 0$;*
3. *$f \in L^2\left(0,T;L^2(\Omega)\right)$, $g \in H^1_0(\Omega)$, and $h \in L^2(\Omega)$.*

Then $a(u,v;t) = a(v,u;t)$ is a symmetric bilinear form on $H^1_0(\Omega)$ Moreover, there exist constants $C > 0$, $\beta > 0$, and $\gamma \in \mathbb{R}$ such that for every $u, v \in H^1_0(\Omega)$

$$
\begin{aligned}
\beta\|u\|^2_{H^1_0} &\le a(u,u;t) + \gamma\|u\|^2_{L^2},\\
|a(u,v;t)| &\le C\,\|u\|_{H^1_0}\,\|v\|_{H^1_0}.\\
|a_t(u,v;t)| &\le C\,\|u\|_{H^1_0}\,\|v\|_{H^1_0}.
\end{aligned}
\tag{7.7}
$$

We define weak solutions of (7.5) as follows.

Definition 7.2. A function $u : [0,T] \to H^1_0(\Omega)$ is a weak solution of (7.5) if:

(1) u has weak derivatives u_t and u_{tt} and

$$
u \in C\left([0,T];H^1_0(\Omega)\right), \quad u_t \in C\left([0,T];L^2(\Omega)\right), \quad u_{tt} \in L^2\left(0,T;H^{-1}(\Omega)\right);
$$

(2) For every $v \in H^1_0(\Omega)$,

$$
\langle u_{tt}(t), v\rangle + a\left(u(t), v; t\right) = \left(f(t), v\right)_{L^2} \tag{7.8}
$$

for t pointwise a.e. in $[0,T]$ where a is defined in (7.6);

(3) $u(0) = g$ and $u_t(0) = h$.

We then have the following existence result.

Theorem 7.3. *Suppose that the conditions in Assumption 7.1 are satisfied. Then for every $f \in L^2\left(0,T;L^2(\Omega)\right)$, $g \in H^1_0(\Omega)$, and $h \in L^2(\Omega)$, there is a unique weak solution of (7.5), in the sense of Definition 7.2. Moreover, there is a constant C, depending only on Ω, T, and the coefficients of L, such that*

$$
\begin{aligned}
&\|u\|_{L^\infty(0,T;H^1_0)} + \|u_t\|_{L^\infty(0,T;L^2)} + \|u_{tt}\|_{L^2(0,T;H^{-1})}\\
&\qquad\qquad \le C\left(\|f\|_{L^2(0,T;L^2)} + \|g\|_{H^1_0} + \|h\|_{L^2}\right).
\end{aligned}
$$

7.3. Existence of weak solutions

We prove an existence result in this section. The continuity and uniqueness of weak solutions is proved in the next sections.

7.3.1. Construction of approximate solutions. As for the Galerkin approximation of the heat equation, let E_N be the N-dimensional subspace of $H_0^1(\Omega)$ given in (6.15)–(6.16) and P_N the orthogonal projection onto E_N given by (6.17).

Definition 7.4. A function $u_N : [0,T] \to E_N$ is an approximate solution of (7.5) if:

(1) $u_N \in L^2(0,T;E_N)$, $u_{Nt} \in L^2(0,T;E_N)$, and $u_{Ntt} \in L^2(0,T;E_N)$;
(2) for every $v \in E_N$

$$(u_{Ntt}(t), v)_{L^2} + a\left(u_N(t), v; t\right) = \left(f(t), v\right)_{L^2} \tag{7.9}$$

pointwise *a.e.* in $t \in (0,T)$;
(3) $u_N(0) = P_N g$, and $u_{Nt}(0) = P_N h$.

Since $u_N \in H^2(0,T;E_N)$, it follows from the Sobolev embedding theorem for functions of a single variable t that $u_N \in C^1([0,T];E_N)$, so the initial condition (3) makes sense. Equation (7.9) is equivalent to an $N\times N$ linear system of second-order ODEs with coefficients that are L^∞ functions of t. By standard ODE theory, it has a solution $u_N \in H^2(0,T;E_N)$; if $a(w_j, w_k; t)$ and $\left(f(t), w_j\right)_{L^2}$ are continuous functions of time, then $u_N \in C^2(0,T;E_N)$. Thus, we have the following existence result.

Proposition 7.5. *For every $N \in \mathbb{N}$, there exists a unique approximate solution $u_N : [0,T] \to E_N$ of (7.5) with*

$$u_N \in C^1\left([0,T];E_N\right), \qquad u_{Ntt} \in L^2\left(0,T;E_N\right).$$

7.3.2. Energy estimates for approximate solutions. The derivation of energy estimates for the approximate solutions follows the derivation of the *a priori* energy estimates for the wave equation.

Proposition 7.6. *There exists a constant C, depending only on T, Ω, and the coefficient functions a^{ij}, c, such that for every $N \in \mathbb{N}$ the approximate solution u_N given by Proposition 7.5 satisfies*

$$\begin{aligned} \|u_N\|_{L^\infty(0,T;H_0^1)} + \|u_{Nt}\|_{L^\infty(0,T;L^2)} + \|u_{Ntt}\|_{L^2(0,T;H^{-1})} \\ \leq C\left(\|f\|_{L^2(0,T;L^2)} + \|g\|_{H_0^1} + \|h\|_{L^2}\right). \end{aligned} \tag{7.10}$$

Proof. Taking $v = u_{Nt}(t) \in E_N$ in (7.9), we find that

$$\left(u_{Ntt}(t), u_{Nt}(t)\right)_{L^2} + a\left(u_N(t), u_{Nt}(t); t\right) = \left(f(t), u_{Nt}(t)\right)_{L^2}$$

pointwise *a.e.* in $(0,T)$. Since a is symmetric, it follows that

$$\frac{1}{2}\frac{d}{dt}\left[\|u_{Nt}\|_{L^2}^2 + a\left(u_N, u_N; t\right)\right] = \left(f, u_{Nt}\right)_{L^2} + a_t\left(u_N, u_N; t\right).$$

Integrating this equation with respect to t, we get

$$\begin{aligned}
&\|u_{Nt}\|_{L^2}^2 + a\,(u_N, u_N; t) \\
&\qquad = 2\int_0^t \left[(f, u_{Ns})_{L^2} + a_s\,(u_N, u_N; s)\right]\,ds + a\,(P_N g, P_N g; 0) + \|P_N h\|_{L^2}^2 \\
&\qquad \le \int_0^t \left(\|u_{Ns}\|_{L^2}^2 + C\,\|u_N\|_{H_0^1}^2\right)\,ds + \|f\|_{L^2(0,T;L^2)}^2 + C\,\|g\|_{H_0^1}^2 + \|h\|_{L^2}^2\,,
\end{aligned}$$

where we have used (7.7), the fact that $\|P_N h\|_{L^2} \le \|h\|$, $\|P_N g\|_{H_0^1} \le \|g\|_{H_0^1}$, and the inequality

$$\begin{aligned}
2\int_0^t (f, u_{Ns})_{L^2} &\le 2\left(\int_0^t \|f\|_{L^2}^2\,ds\right)^{1/2}\left(\int_0^t \|u_{Ns}\|_{L^2}^2\,ds\right)^{1/2} \\
&\le \int_0^t \|u_{Ns}\|_{L^2}^2\,ds + \int_0^T \|f\|_{L^2}^2\,ds.
\end{aligned}$$

Using the uniform ellipticity condition in (7.7) to estimate $\|u_N\|_{H_0^1}^2$ in terms of $a(u_N, u_N; t)$ and a lower L^2-norm of u_N, we get for $0 \le t \le T$ that

$$\begin{aligned}
\|u_{Nt}\|_{L^2}^2 + \beta\,\|u_N\|_{H_0^1}^2 \le \int_0^t \left(\|u_{Ns}\|_{L^2}^2 + C\,\|u_N\|_{H_0^1}^2\right)\,ds + \gamma\,\|u_N\|_{L^2}^2 \\
+ \|f\|_{L^2(0,T;L^2)}^2 + C\,\|g\|_{H_0^1}^2 + \|h\|_{L^2}^2\,.
\end{aligned} \tag{7.11}$$

We estimate the L^2-norm of u_N by

$$\begin{aligned}
\|u_N\|_{L^2}^2 &= 2\int_0^t (u_N, u_N)_{L^2}\,ds + \|P_N g\|_{L^2}^2 \\
&\le 2\left(\int_0^t \|u_N\|_{L^2}^2\,ds\right)^{1/2}\left(\int_0^t \|u_{Ns}\|_{L^2}^2\,ds\right)^{1/2} + \|g\|_{L^2}^2 \\
&\le \int_0^t \left(\|u_N\|_{L^2}^2 + \|u_{Ns}\|_{L^2}^2\right)\,ds + \|g\|_{L^2}^2 \\
&\le \int_0^t \left(\|u_{Ns}\|_{L^2}^2 + C\,\|u_N\|_{H_0^1}^2\right)\,ds + C\|g\|_{H_0^1}^2.
\end{aligned}$$

Using this result in (7.11), we find that

$$\begin{aligned}
\|u_{Nt}\|_{L^2}^2 + \|u_N\|_{H_0^1}^2 \le C_1\int_0^t \left(\|u_{Ns}\|_{L^2}^2 + \|u_N\|_{H_0^1}^2\right)\,ds \\
+ C_2\left(\|f\|_{L^2(0,T;L^2)}^2 + \|g\|_{H_0^1}^2 + \|h\|_{L^2}^2\right)
\end{aligned} \tag{7.12}$$

for some constants $C_1, C_2 > 0$. Thus, defining $E : [0, T] \to \mathbb{R}$ by

$$E = \|u_{Nt}\|_{L^2}^2 + \|u_N\|_{H_0^1}^2\,,$$

we have

$$E(t) \le C_1\int_0^t E(s)\,ds + C_2\left(\|f\|_{L^2(0,T;L^2)}^2 + \|h\|_{L^2}^2 + \|g\|_{H_0^1}^2\right).$$

Gronwall's inequality (Lemma 1.47) implies that

$$E(t) \le C_2\left(\|f\|_{L^2(0,T;L^2)}^2 + \|h\|_{L^2}^2 + \|g\|_{H_0^1}^2\right)e^{C_1 t},$$

and we conclude that there is a constant C such that

$$\sup_{[0,T]}\left(\|u_{Nt}\|_{L^2}^2+\|u_N\|_{H_0^1}^2\right)\le C\left(\|f\|_{L^2(0,T;L^2)}^2+\|h\|_{L^2}^2+\|g\|_{H_0^1}^2\right). \tag{7.13}$$

Finally, from the Galerkin equation (7.9), we have for every $v\in E_N$ that

$$(u_{Ntt},v)_{L^2}=(f,v)_{L^2}-a\,(u_N,v;t)$$

pointwise *a.e.* in t. Since $u_{Ntt}\in E_N$, it follows that

$$\|u_{Ntt}\|_{H^{-1}}=\sup_{v\in E_N\setminus\{0\}}\frac{(u_{Ntt},v)_{L^2}}{\|v\|_{H_0^1}}\le C\left(\|f\|_{L^2}+\|u_N\|_{H_0^1}\right).$$

Squaring this inequality, integrating with respect to t, and using (7.10) we get

$$\begin{aligned}\int_0^T\|u_{Ntt}\|_{H^{-1}}^2\,dt&\le C\int_0^T\left(\|f\|_{L^2}^2+\|u_N\|_{H_0^1}^2\right)dt\\&\le C\left(\|f\|_{L^2(0,T;L^2)}^2+\|h\|_{L^2}^2+\|g\|_{H_0^1}^2\right).\end{aligned} \tag{7.14}$$

Combining (7.13)–(7.14), we get (7.10). □

7.3.3. Convergence of approximate solutions. The uniform estimates for the approximate solutions allows us to obtain a weak solution as the limit of a subsequence of approximate solutions in an appropriate weak-star topology. We use a weak-star topology because the estimates are L^∞ in time, and L^∞ is not reflexive. From Theorem 6.30, if X is reflexive Banach space, such as a Hilbert space, then

$$u_N\overset{*}{\rightharpoonup}u\qquad\text{in }L^\infty(0,T;X)$$

if and only if

$$\int_0^T\langle u_N(t),w(t)\rangle\,dt\to\int_0^T\langle u(t),w(t)\rangle\,dt\qquad\text{for every }w\in L^1(0,T;X').$$

Theorem 1.19 then gives us weak-star compactness of the approximations and convergence of a subsequence as stated in the following proposition.

Proposition 7.7. *There is a subsequence $\{u_N\}$ of approximate solutions and a function u with such that*

$$\begin{aligned}u_N&\overset{*}{\rightharpoonup}u&&\text{as }N\to\infty\text{ in }L^\infty\left(0,T;H_0^1\right),\\u_{Nt}&\overset{*}{\rightharpoonup}u_t&&\text{as }N\to\infty\text{ in }L^\infty\left(0,T;L^2\right),\\u_{Ntt}&\rightharpoonup u_{tt}&&\text{as }N\to\infty\text{ in }L^2\left(0,T;H^{-1}\right),\end{aligned}$$

where u satisfies (7.8).

Proof. By Proposition 7.6, the approximate solutions $\{u_N\}$ are uniformly bounded in $L^\infty(0,T;H_0^1)$, and their time-derivatives are uniformly bounded in $L^\infty(0,T;L^2)$. It follows from the Banach-Alaoglu theorem, and the usual argument that a weak limit of derivatives is the derivative of the weak limit, that there is a subsequence of approximate solutions, which we still denote by $\{u_N\}$, such that

$$u_N\overset{*}{\rightharpoonup}u\quad\text{in }L^\infty(0,T;H_0^1),\qquad u_{Nt}\overset{*}{\rightharpoonup}u_t\quad\text{in }L^\infty(0,T;L^2).$$

Moreover, since $\{u_{Ntt}\}$ is uniformly bounded in $L^2(0,T;H^{-1})$, we can choose the subsequence so that

$$u_{Ntt}\rightharpoonup u_{tt}\qquad\text{in }L^2(0,T;H^{-1}).$$

Thus, the weak-star limit u satisfies

$$u \in L^{\infty}(0,T;H_0^1), \quad u_t \in L^{\infty}(0,T;L^2), \quad u_{tt} \in L^2(0,T;H^{-1}). \tag{7.15}$$

Passing to the limit $N \to \infty$ in the Galerkin equations(7.9), we find that u satisfies (7.8) for every $v \in H_0^1(\Omega)$. In detail, consider time-dependent test functions of the form $w(t) = \phi(t)v$ where $\phi \in C_c^{\infty}(0,T)$ and $v \in E_M$, as for the parabolic equation. Multiplying (7.9) by $\phi(t)$ and integrating the result with respect to t, we find that for $N \geq M$

$$\int_0^T (u_{Ntt}, w)_{L^2}\ dt + \int_0^T a\left(u_N, w; t\right)\ dt = \int_0^T (f, w)_{L^2}\ dt.$$

Taking the limit of this equation as $N \to \infty$, we get

$$\int_0^T (u_{tt}, w)_{L^2}\ dt + \int_0^T a\left(u, w; t\right)\ dt = \int_0^T (f, w)_{L^2}\ dt.$$

By density, this equation holds for $w(t) = \phi(t)v$ where $v \in H_0^1(\Omega)$, and then since $\phi \in C_c^{\infty}(0,T)$ is arbitrary, we get(7.9). □

7.4. Continuity of weak solutions

In this section, we show that the weak solutions obtained above satisfy the continuity requirement (1) in Definition 7.2. To do this, we show that u and u_t are weakly continuous with values in H_0^1, and L^2 respectively, then use the energy estimate to show that the 'energy' $E : (0,T) \to \mathbb{R}$ defined by

$$E = \|u_t\|_{L^2} + a(u,u;t) \tag{7.16}$$

is a continuous function of time. This gives continuity in norm, which together with weak continuity implies strong continuity. The argument is essentially the same as the proof that if a sequence $\{x_n\}$ converges weakly to x in a Hilbert space $\mathcal{H}$ and the norms also converge, then the sequence converges strongly:

$$(x - x_n, x - x_n) = \|x\|^2 - 2(x, x_n) + \|x_n\|^2 \to \|x\|^2 - 2(x,x) + \|x\|^2 = 0.$$

See (7.23) below for the analogous formula in this argument.

We begin by proving the weak continuity, which follows from the next lemma.

Lemma 7.8. *Suppose that $\mathcal{V}$, $\mathcal{H}$ are Hilbert spaces and $\mathcal{V} \hookrightarrow \mathcal{H}$ is densely and continuously embedded in $\mathcal{H}$. If*

$$u \in L^{\infty}\left(0,T;\mathcal{V}\right), \qquad u_t \in L^2\left(0,T;\mathcal{H}\right),$$

then $u \in C_w\left([0,T];\mathcal{V}\right)$ is weakly continuous.

Proof. We have $u \in H^1\left(0,T;\mathcal{H}\right)$ and the Sobolev embedding theorem, Theorem 6.38, implies that $u \in C\left([0,T];\mathcal{H}\right)$. Let $\omega \in \mathcal{V}'$, and choose $\omega_n \in \mathcal{H}$ such that $\omega_n \to \omega$ in $\mathcal{V}'$. Then

$$|\langle \omega_n, u(t)\rangle - \langle \omega, u(t)\rangle| = |\langle \omega_n - \omega, u(t)\rangle| \leq \|\omega_n - \omega\|_{\mathcal{V}'}\, \|u(t)\|_{\mathcal{V}}.$$

Thus,

$$\sup_{[0,T]} |\langle \omega_n, u\rangle - \langle \omega, u\rangle| \leq \|\omega_n - \omega\|_{\mathcal{V}'}\, \|u\|_{L^{\infty}(0,T;\mathcal{V})} \to 0 \qquad \text{as } n \to \infty,$$

so $\langle \omega_n, u\rangle$ converges uniformly to $\langle \omega, u\rangle$. Since $\langle \omega_n, u\rangle \in C\left([0,T];\mathcal{V}\right)$ it follows that $\langle \omega, u\rangle \in C\left([0,T];\mathcal{V}\right)$, meaning that u is weakly continuous into $\mathcal{V}$. □

Lemma 7.9. *Let u be a weak solution constructed in Proposition 7.7. Then*

$$u \in C_w([0,T]; H_0^1), \qquad u_t \in C_w([0,T]; L^2) \tag{7.17}$$

Proof. This follows at once from Lemma 7.9 and the fact that

$$u \in L^\infty\left(0,T; H_0^1\right), \quad u_t \in L^\infty\left(0,T; H^{-1}\right), \quad u_{tt} \in L^2\left(0,T; H^{-1}\right),$$

where $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$. □

Next, we prove that the energy is continuous. In doing this, we have to be careful not to assume more regularity in time that we know.

Lemma 7.10. *Suppose that L is given by (7.4) and a by (7.6), where the coefficients satisfy the conditions in Assumption 7.1. If*

$$u \in L^2\left(0,T; H_0^1(\Omega)\right), \quad u_t \in L^2\left(0,T; L^2(\Omega)\right), \quad u_{tt} \in L^2\left(0,T; H^{-1}(\Omega)\right),$$

and

$$u_{tt} + Lu \in L^2\left(0,T; L^2(\Omega)\right), \tag{7.18}$$

then

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t\|_{L^2}^2 + a(u,u;t)\right) = (u_{tt} + Lu, u_t)_{L^2} + \frac{1}{2}a_t(u,u;t). \tag{7.19}$$

and $E : (0,T) \to \mathbb{R}$ defined in (7.16) is an absolutely continuous function.

Proof. We show first that (7.19) holds in the sense of (real-valued) distributions on $(0,T)$. The relation would be immediate if u was sufficiently smooth to allow us to expand the derivatives with respect to t. We prove it for general u by mollification.

It is sufficient to show that (7.19) holds in the distributional sense on compact subsets of $(0,T)$. Let $\zeta \in C_c^\infty(\mathbb{R})$ be a cut-off function that is equal to one on some subinterval $I \Subset (0,T)$ and zero on $\mathbb{R} \setminus (0,T)$. Extend u to a compactly supported function $\zeta u : \mathbb{R} \to H_0^1(\Omega)$, and mollify this function with the standard mollifier $\eta^\epsilon : \mathbb{R} \to \mathbb{R}$ to obtain

$$u^\epsilon = \eta^\epsilon * (\zeta u) \in C_c^\infty\left(\mathbb{R}; H_0^1\right).$$

Mollifying (7.18), we also have that

$$u_{tt}^\epsilon + Lu^\epsilon \in L^2\left(\mathbb{R}; L^2\right). \tag{7.20}$$

Without (7.18), we would only have $Lu^\epsilon \in L^2\left(\mathbb{R}; H^{-1}\right)$.

Since u^ϵ is a smooth, H_0^1-valued function and a is symmetric, we have that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|u_t^\epsilon\|_{L^2}^2 + a\left(u^\epsilon, u^\epsilon; t\right)\right) &= \langle u_{tt}^\epsilon, u_t^\epsilon\rangle + a\left(u^\epsilon, u_t^\epsilon; t\right) + \frac{1}{2}a_t\left(u^\epsilon, u^\epsilon; t\right)\\
&= \langle u_{tt}^\epsilon, u_t^\epsilon\rangle + \langle Lu^\epsilon, u_t^\epsilon\rangle + \frac{1}{2}a_t\left(u^\epsilon, u^\epsilon; t\right)\\
&= \langle u_{tt}^\epsilon + Lu^\epsilon, u_t^\epsilon\rangle + \frac{1}{2}a_t\left(u^\epsilon, u^\epsilon; t\right)\\
&= \left(u_{tt}^\epsilon + Lu^\epsilon, u_t^\epsilon\right)_{L^2} + \frac{1}{2}a_t\left(u^\epsilon, u^\epsilon; t\right).
\end{aligned} \tag{7.21}$$

Here, we have used (7.20) and the identity

$$a(u,v;t) = \langle L(t)u, v\rangle \qquad \text{for } u, v \in H_0^1.$$

Note that we cannot use this identity to rewrite $a(u, u_t; t)$ if u is the unmollified function, since we know only that $u_t \in L^2$. Taking the limit of (7.21) as $\epsilon \to 0^+$, we get the same equation for ζu, and hence (7.19) holds on every compact subinterval of $(0,T)$, which proves the equation.

The right-hand side of (7.19) belongs to $L^1(0,T)$ since

$$\begin{aligned}\int_0^T \left(u_{tt} + Lu, u_t\right)_{L^2}\, dt &\leq \int_0^T \|u_{tt} + Lu\|_{L^2} \|u_t\|_{L^2}\, dt \\ &\leq \|u_{tt} + Lu\|_{L^2(0,T;L^2)} \|u_t\|_{L^2(0,T;L^2)}, \\ \int_0^T a_t\left(u,u;t\right)\, dt &\leq \int_0^T C\|u\|_{H_0^1}^2\, dt \\ &\leq C\|u\|_{L^2(0,T;H_0^1)}^2.\end{aligned}$$

Thus, E in (7.16) is the integral of an L^1-function, so it is absolutely continuous. □

Proposition 7.11. *Let u be a weak solution constructed in Proposition 7.7. Then*

$$u \in C\left([0,T]; H_0^1(\Omega)\right), \qquad u_t \in C\left([0,T]; L^2(\Omega)\right). \tag{7.22}$$

Proof. Using the weak continuity of u, u_t from Lemma 7.9, the continuity of E from Lemma 7.10, energy, and the continuity of a_t on H_0^1, we find that as $t \to t_0$,

$$\begin{aligned}&\|u_t(t) - u_t(t_0)\|_{L^2}^2 + a\left(u(t) - u(t_0), u(t) - u(t_0); t_0\right) \\ &\quad = \|u_t(t)\|_{L^2}^2 - 2\left(u_t(t), u_t(t_0)\right)_{L^2} + \|u_t(t_0)\|_{L^2}^2 \\ &\qquad + a\left(u(t), u(t); t_0\right) - 2a\left(u(t), u(t_0); t_0\right) + a\left(u(t_0), u(t_0); t_0\right) \\ &\quad = \|u_t(t)\|_{L^2} + a\left(u(t), u(t); t\right) + \|u_t(t_0)\|_{L^2} + a\left(u(t_0), u(t_0); t_0\right) \\ &\qquad + a\left(u(t), u(t); t_0\right) - a\left(u(t), u(t); t\right) \\ &\qquad - 2\left(u_t(t), u_t(t_0)\right)_{L^2} - 2a\left(u(t), u(t_0); t_0\right) \\ &\quad = E(t) + E(t_0) + a\left(u(t), u(t); t_0\right) - a\left(u(t), u(t); t\right) \\ &\qquad - 2\left\{\left(u_t(t), u_t(t_0)\right)_{L^2} + a\left(u(t), u(t_0); t_0\right)\right\} \\ &\quad \to E(t_0) + E(t_0) - 2\left\{\|u_t(t_0)\|_{L^2} + a\left(u(t_0), u(t_0); t_0\right)\right\} = 0.\end{aligned} \tag{7.23}$$

Finally, using this result, the coercivity estimate

$$\theta\|u(t) - u(t_0)\|_{H_0^1}^2 \leq a\left(u(t) - u(t_0), u(t) - u(t_0); t_0\right) + \gamma\|\, \|u(t) - u(t_0)\|_{L^2}^2$$

and the fact that $u \in C\left(0,T; L^2\right)$ by Sobolev embedding, we conclude that

$$\lim_{t\to t_0} \|u_t(t) - u_t(t_0)\|_{L^2} = 0, \qquad \lim_{t\to t_0} \|u(t) - u(t_0)\|_{H_0^1} = 0,$$

which proves (7.22). □

This completes the proof of the existence of a weak solution in the sense of Definition 7.2.

7.5. Uniqueness of weak solutions

The proof of uniqueness of weak solutions of the IBVP (7.5) for the second-order hyperbolic PDE requires a more careful argument than for the corresponding parabolic IBVP. To get an energy estimate in the parabolic case, we use the test function $v = u(t)$; this is permissible since $u(t) \in H_0^1(\Omega)$. To get an estimate in the hyperbolic case, we would like to take $v = u_t(t)$, but we cannot do this directly, since we know only that $u_t(t) \in L^2(\Omega)$. Instead we fix $t_0 \in (0,T)$ and use as a test function

$$(7.24) \qquad \begin{aligned} v(t) &= \int_{t_0}^{t} u(s)\,ds \qquad &&\text{for } 0 < t \le t_0, \\ v(t) &= 0 &&\text{for } t_0 < t < T. \end{aligned}$$

To motivate this choice, consider an *a priori* estimate for the wave equation. Suppose that

$$u_{tt} = \Delta u, \qquad u(0) = u_t(0) = 0.$$

Multiplying the PDE by v in (7.24), and using the fact that $v_t = u$ we get for $0 < t < t_0$ that

$$\left(vu_t - \frac{1}{2}u^2 + \frac{1}{2}|Dv|^2\right)_t - \operatorname{div}(vDu) = 0.$$

We integrate this equation over Ω to get

$$\frac{d}{dt}\int_\Omega \left(vu_t - \frac{1}{2}u^2 + \frac{1}{2}|Dv|^2\right)\,dx = 0$$

The boundary terms $vDu\cdot\nu$ vanish since $u = 0$ on $\partial\Omega$ implies that $v = 0$. Integrating this equation with respect to t over $(0, t_0)$, and using the fact that $u = u_t = 0$ at $t = 0$ and $v = 0$ at $t = t_0$, we find that

$$\|u\|_{L^2}^2(t_0) + \|v\|_{H_0^1}^2(0) = 0.$$

Since this holds for every $t_0 \in (0,T)$, we conclude that $u = 0$.

The proof of the next proposition is the same calculation for weak solutions.

Proposition 7.12. *A weak solution of (7.5) in the sense of Definition 7.2 is unique.*

Proof. Since the equation is linear, to show uniqueness it is sufficient to show that the only solution u of (7.5) with zero data ($f = 0$, $g = 0$, $h = 0$) is $u = 0$.

Let $v \in C\left([0,T]; H_0^1\right)$ be given by (7.24). Using $v(t)$ in (7.8), we get for $0 < t < t_0$ that

$$\langle u_{tt}(t), v(t)\rangle + a\left(u(t), v(t); t\right) = 0.$$

Since $u = v_t$ and a is a symmetric bilinear form on H_0^1, it follows that

$$\frac{d}{dt}\left[(u_t, v)_{L^2} - \frac{1}{2}(u,u)_{L^2} + \frac{1}{2}a(v,v;t)\right] = a_t(v,v;t).$$

Integrating this equation from 0 to t_0, and using the fact that

$$u(0) = 0, \qquad u_t(0) = 0, \qquad v(t_0) = 0,$$

we get

$$\|u(t_0)\|_{L^2} + a\left(v(0), v(0); 0\right) = -2\int_0^{t_0} a(v,v;t)\,dt.$$

Using the coercivity and boundedness estimates for a in (7.7), we find that

$$(7.25) \qquad \|u(t_0)\|_{L^2}^2 + \beta\|v(0)\|_{H_0^1}^2 \le C\int_0^{t_0}\|v(t)\|_{H_0^1}^2\,dt + \gamma\|v(0)\|_{L^2}^2.$$

Writing $w(t) = -v(t_0 - t)$ for $0 < t < t_0$, we have from (7.24) that

$$w(t) = -\int_{t_0}^{t_0 - t} u(s)\,ds = \int_0^t u(t_0 - s)\,ds$$

and

$$v(0) = -w(t_0) = -\int_0^{t_0} u(t_0 - s)\, ds = \int_0^{t_0} u(t)\, dt,$$

$$\int_0^{t_0} \|v(t)\|_{H_0^1}^2 \, dt = \int_0^{t_0} \|w(t_0 - t)\|_{H_0^1}^2 \, dt = \int_0^{t_0} \|w(t)\|_{H_0^1}^2 \, dt.$$

Using these expressions in (7.25), we get an estimate of the form

$$\|u(t_0)\|_{L^2}^2 + \|w(t_0)\|_{H_0^1}^2 \leq C \int_0^{t_0} \left(\|u(t)\|_{L^2}^2 + \|w(t)\|_{H_0^1}^2 \right) dt$$

for every $0 < t_0 < T$. Since $u(0) = 0$ and $w(0) = 0$, Gronwall's inequality implies that u, w are zero on $[0, T]$, which proves the uniqueness of weak solutions. □

This proposition completes the proof of Theorem 7.3. For the regularity theory of these weak solutions see §7.2.3 of [**8**].