CHAPTER 8

Friedrich symmetric systems

In this chapter, we describe a theory due to Friedrich [**13**] for positive symmetric systems, which gives the existence and uniqueness of weak solutions of boundary value problems under appropriate positivity conditions on the PDE and the boundary conditions. No assumptions about the type of the PDE are required, and the theory applies equally well to hyperbolic, elliptic, and mixed-type systems.

8.1. A BVP for symmetric systems

Let Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. Consider a BVP for an $m \times m$ system of PDEs for $u : \Omega \to \mathbb{R}^m$ of the form

$$\begin{aligned} &A^i \partial_i u + Cu = f && \text{in } \Omega, \\ &B_- u = 0 && \text{on } \partial\Omega, \end{aligned} \tag{8.1}$$

where A^i, C, B_- are $m \times m$ coefficient matrices, $f : \Omega \to \mathbb{R}^m$, and we use the summation convention. We assume throughout that A^i is symmetric.

We define a boundary matrix on $\partial\Omega$ by

$$B = \nu_i A^i \tag{8.2}$$

where ν is the outward unit normal to $\partial\Omega$. We assume that the boundary is non-characteristic and that (8.1) satisfies the following smoothness conditions.

Definition 8.1. The BVP (8.1) is smooth if:

(1) The domain Ω is bounded and has C^2-boundary.
(2) The symmetric matrices $A^i : \overline{\Omega} \to \mathbb{R}^{m\times m}$ are continuously differentiable on the closure $\overline{\Omega}$, and $C : \overline{\Omega} \to \mathbb{R}^{m\times m}$ is continuous on $\overline{\Omega}$.
(3) The boundary matrix $B_- : \partial\Omega \to \mathbb{R}^{m\times m}$ is continuous on $\partial\Omega$.

These assumptions can be relaxed, but our goal is to describe the theory in its basic form with a minimum of technicalities.

Let L denote the operator in (8.1) and L^* its formal adjoint,

$$L = A^i \partial_i + C, \qquad L^* = -A^i \partial_i + C^T - \partial_i A^i. \tag{8.3}$$

For brevity, we write spaces of continuously differentiable and square integrable vector-valued functions as

$$C^1(\overline{\Omega}) = C^1(\overline{\Omega}; \mathbb{R}^m), \qquad L^2(\Omega) = L^2(\Omega; \mathbb{R}^m),$$

with a similar notation for matrix-valued functions.

Proposition 8.2 (Green's identity). *If the smoothness assumptions in Definition 8.1 are satisfied and $u, v \in C^1(\overline{\Omega})$, then*

$$\int_\Omega v^T L u \, dx - \int_\Omega u^T L^* v \, dx = \int_{\partial\Omega} v^T B u \, dS, \tag{8.4}$$

where B is defined in (8.2).

PROOF. Using the symmetry of A^i, we have

$$v^T Lu = u^T L^* v + \partial_i \left(v^T A^i u \right).$$

The result follows by integration and the use of Green's theorem. □

The smoothness assumptions are sufficient to ensure that Green's theorem applies, although it also holds under weaker assumptions.

PROPOSITION 8.3 (Energy identity). *If the smoothness assumptions in Definition 8.1 are satisfied and $u \in C^1(\overline{\Omega})$, then*

$$\int_\Omega u^T \left(C + C^T - \partial_i A^i \right) u \, dx + \int_{\partial\Omega} u^T B u \, dS = 2 \int_\Omega f^T u \, dx \tag{8.5}$$

where B is defined in (8.2) and $Lu = f$.

PROOF. Taking the inner product of the equation $Lu = f$ with u, adding the transposed equation, and combining the derivatives of u, we get

$$\partial_i \left(u^T A^i u \right) + u^T \left(C + C^T - \partial_i A^i \right) u = 2 f^T u.$$

The result follows by integration and the use of Green's theorem. □

To get energy estimates, we want to ensure that the volume integral in (8.5) is positive, which leads to the following definition.

DEFINITION 8.4. The system in (8.1) is a positive symmetric system if the matrices A^i are symmetric and there exists a constant $c > 0$ such that

$$C + C^T - \partial_i A^i \geq 2cI. \tag{8.6}$$

8.2. Boundary conditions

We assume that the domain has non-characteristic boundary, meaning that the boundary matrix $B = \nu_i A^i$ is nonsingular on $\partial\Omega$. The analysis extends to characteristic boundaries with constant multiplicity, meaning that the rank of B is constant on $\partial\Omega$ [**25, 34**].

To get estimates, we need the boundary terms in (8.5) to be positive for all u such that $B_- u = 0$. Furthermore, to get estimates for the adjoint problem, we need the adjoint boundary terms to be negative. This is the case if the boundary conditions are maximally positive in the following sense [**13**].

DEFINITION 8.5. Let $B = \nu_i A^i$ be a nonsingular, symmetric boundary matrix. A boundary condition $B_- u = 0$ on $\partial\Omega$ is maximally positive if there is a (not necessarily symmetric) matrix function $M : \partial\Omega \to \mathbb{R}^{m \times m}$ such that:

(1) $B = B_+ + B_-$ where $B_+ = B + M$, and $B_- = B - M$;
(2) $M + M^T \geq 0$ (positivity);
(3) $\mathbb{R}^m = \ker B_+ \oplus \ker B_-$ (maximality).

The adjoint boundary condition to $B_- u = 0$ is $B_+^T v = 0$, as can be seen from the decomposition

$$v^T B u = u^T B_+^T v + v^T B_- u.$$

If $B_- u = 0$ on $\partial\Omega$, then

$$u^T B u = u^T \left(B_+ - B_- \right) u = u^T \left(M + M^T \right) u \geq 0, \tag{8.7}$$

while if $B_+^T v = 0$ on $\partial\Omega$, then

$$v^T B v = v^T\left(-B_+ + B_-\right) v = -v^T\left(M + M^T\right) v \leq 0. \tag{8.8}$$

The boundary condition $B_- u = 0$ can also be formulated as: $u \in \mathcal{N}_+$ where $\mathcal{N}_+ = \ker B_-$ is a family of subspaces defined on $\partial\Omega$. An equivalent way to state Definition 8.5 is that the subspace $\mathcal{N}_+$ is a maximally positive subspace for B, meaning that B is positive (≥ 0) on $\mathcal{N}_+$ and not positive on any strictly larger subspace of $\mathbb{R}^m$ that contains $\mathcal{N}_+$.

The adjoint boundary condition $B_+^T v = 0$ may be written as $v \in \mathcal{N}_-$ where $\mathcal{N}_- = \ker B_+^T$ is a maximally negative subspace that complements $\mathcal{N}_+$, and

$$\mathbb{R}^m = \mathcal{N}_+ \oplus \mathcal{N}_-, \qquad \mathcal{N}_+ = \left(B\mathcal{N}_-\right)^\perp, \qquad \mathcal{N}_- = \left(B\mathcal{N}_+\right)^\perp.$$

We may consider $\mathbb{R}^m$ as a vector space with an indefinite inner product given by $\langle u, v\rangle = u^T B v$. It follows from standard results about indefinite inner product spaces that if $\mathbb{R}^m = \mathcal{N}_+ \oplus \mathcal{N}_-$ where $\mathcal{N}_+$ is a maximally positive subspace for $\langle\cdot,\cdot\rangle$, then $\mathcal{N}_-$ is a maximally negative subspace. Moreover, the dimension of $\mathcal{N}_+$ is equal to the number of positive eigenvalues of B, and the dimension of $\mathcal{N}_-$ is equal to the number of negative eigenvalues of B. In particular, the dimensions of $\mathcal{N}_\pm$ are constant on each connected component of $\partial\Omega$ if B is continuous and non-singular.

8.3. Uniqueness of smooth solutions

Under the above positivity assumptions, we can estimate a smooth solution u of (8.1) by the right-hand side f. A similar result holds for the adjoint problem. Let

$$\|u\| = \left(\int_\Omega |u|^2\, dx\right)^{1/2}, \qquad (u, v) = \int_\Omega u^T v\, dx \tag{8.9}$$

denote the standard L^2-norm and inner product, where $|u|$ denotes the Euclidean norm of $u \in \mathbb{R}^m$.

THEOREM 8.6. *Let L, L^* denote the operators in (8.3), and suppose that the smoothness conditions in Definition 8.1 and the positivity conditions in Definition 8.4, Definition 8.5 are satisfied. If $u \in C^1(\overline{\Omega})$ and $B_- u = 0$ on $\partial\Omega$, then $c\|u\| \leq \|Lu\|$. If $v \in C^1(\overline{\Omega})$ and $B_+^T v = 0$ on $\partial\Omega$, then $c\|v\| \leq \|L^* v\|$.*

PROOF. If $B_- u = 0$, then the energy identity (8.5), the positivity conditions (8.6)–(8.7), and the Cauchy-Schwartz inequality imply that

$$\begin{aligned} 2c\|u\|^2 &\leq \int_\Omega u^T\left(C + C^T - \partial_i A^i\right) u\, dx + \int_{\partial\Omega} u^T B u\, dS \\ &\leq 2\int_\Omega u^T L u\, dx \\ &\leq 2\|u\|\,\|Lu\| \end{aligned}$$

so $c\|u\| \leq \|Lu\|$. Similarly, if $B_+^T v = 0$, then Green's formula and (8.8) imply that

$$2c\|v\|^2 \leq 2\int_\Omega v^T L^* v\, dx - \int_{\partial\Omega} v^T B v\, dS = 2\int_\Omega v^T L^* v\, dx \leq 2\|v\|\,\|L^* v\|,$$

which proves the result for L^*. □

COROLLARY 8.7. *If the smoothness conditions in Definition 8.1 and the positivity conditions in Definition 8.4, Definition 8.5 are satisfied, then a smooth solution* $u \in C^1(\overline{\Omega})$ *of (8.1) is unique.*

PROOF. If u_1, u_2 are two solutions and $u = u_1 - u_2$, then $Lu = 0$ and $B_- u = 0$, so Theorem 8.6 implies that $u = 0$. □

8.4. Existence of weak solutions

We define weak solutions of (8.1) as follows.

DEFINITION 8.8. Let $f \in L^2(\Omega)$. A function $u \in L^2(\Omega)$ is a weak solution of (8.1) if

$$\int_\Omega u^T L^* v\, dx = \int_\Omega f^T v\, dx \qquad \text{for all } v \in D^*,$$

where L^* is the operator defined in (8.3), the space of test functions v is

$$D^* = \left\{ v \in C^1(\overline{\Omega}) : B_+^T v = 0 \text{ on } \partial\Omega \right\}, \tag{8.10}$$

and B_+ is the boundary matrix in Definition 8.5.

It follows from Green's theorem that a smooth function $u \in C^1(\overline{\Omega})$ is a weak solution of (8.1) if and only if it is a classical solution i.e., it satisfies (8.1) pointwise. In general, a weak solution u is a distributional solution of $Lu = f$ in Ω with $u, Lu \in L^2(\Omega)$. The boundary condition $B_- u = 0$ is enforced weakly by the use of test functions v that are not compactly supported in Ω and satisfy the adjoint boundary condition $B_+^T v = 0$.

In particular, functions $u, v \in H^1(\Omega)$ satisfy the integration by parts formula

$$\int_\Omega v^T \partial_i u\, dx = -\int_\Omega u^T \partial_i v\, dx + \int_{\partial\Omega} \nu_i (\gamma v)^T (\gamma u)\, dx'$$

where the trace map

$$\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega) \tag{8.11}$$

is defined by the pointwise evaluation of smooth functions on $\partial\Omega$ extended by density and boundedness to $H^1(\Omega)$.The trace map is not, however, well-defined for general $u \in L^2(\Omega)$.

It follows that if $u \in H^1(\Omega)$ is a weak solution of $Lu = f$, satisfying Definition 8.8, then

$$\int_{\mathbb{R}^{n-1}} v^T \gamma B u\, dx' = 0 \qquad \text{for all } v \in D^*,$$

which implies that $\gamma B_- u = 0$. A similar result holds in a distributional sense if $u, Lu \in L^2(\Omega)$, in which case $\gamma B u \in H^{-1/2}(\partial\Omega)$.

The existence of weak solutions follows immediately from the the Riesz representation theorem and the estimate for the adjoint L^* in Theorem 8.6.

THEOREM 8.9. *If the smoothness conditions in Definition 8.1 and the positivity conditions in Definition 8.4, Definition 8.5 are satisfied, then there is a weak solution* $u \in L^2(\Omega)$ *of (8.1) for every* $f \in L^2(\Omega)$.

PROOF. We write $H = L^2(\Omega)$, where H is equipped with its standard norm and inner product given in (8.9). Let

$$L^* : D^* \subset H \to H$$

where the domain D^* of L^* is given by (8.10), and denote the range of L^* by $W = L^*(D^*) \subset H$. From Theorem 8.6,

$$c\|v\| \leq \|L^* v\| \qquad \text{for all } v \in D^*, \tag{8.12}$$

which implies, in particular, that $L^* : D^* \to W$ is one-to-one.

Define a linear functional $\ell : W \to \mathbb{R}$ by

$$\ell(w) = (f, v) \text{ where } L^* v = w.$$

This functional is well-defined since L^* is one-to-one. Furthermore, ℓ is bounded on W since (8.12) implies that

$$|\ell(w)| \leq \|f\| \, \|v\| \leq \frac{1}{c} \|f\| \, \|w\|.$$

By the Riesz representation theorem, there exists $u \in \overline{W} \subset H$ such that $(u, w) = \ell(w)$ for all $w \in W$, which implies that

$$(u, L^* v) = (f, v) \qquad \text{for all } v \in D^*.$$

This identity it just the statement that u is a weak solution of (8.1). □

8.5. Weak equals strong

A weak solution of (8.1) does not satisfy the same boundary condition as a test function in Definition 8.8. As a result, we cannot derive an energy equation analogous to (8.5) directly from the weak formulation and use it to prove the uniqueness of a weak solution.

To close the gap between the existence of weak solutions and the uniqueness of smooth solutions, we use the fact that weak solutions are strong solutions, meaning that they can be obtained as a limit of smooth solutions.

Definition 8.10. Let $f \in L^2(\Omega)$. A function $u \in L^2(\Omega)$ is a strong solution of (8.1) there exists a sequence of functions $u_n \in C^1(\overline{\Omega})$ such that $B_- u_n = 0$ on $\partial\Omega$ and $u_n \to u$, $L u_n \to f$ in $L^2(\Omega)$ as $n \to \infty$.

In operator-theoretic terms, this definition says that u is a strong solution of (8.1) if the pair $(u, f) \in L^2(\Omega) \times L^2(\Omega)$ belongs to the closure of the graph of the operator

$$\begin{aligned} &L : D \subset L^2(\Omega) \to L^2(\Omega), \\ &D = \left\{ u \in C^1(\overline{\Omega}) : B_- u = 0 \text{ on } \partial\Omega \right\}. \end{aligned}$$

If $\overline{D}$ is the domain of the closure, then

$$\overline{D} \supset \{ u \in H^1(\Omega) : \gamma B_- u = 0 \},$$

but, in general, it is difficult to give an explicit description of $\overline{D}$.

We will prove that a weak solution is a strong solution by mollifying the weak solution. In fact, Friedrichs [**12**] introduced mollifiers for exactly this purpose. The proof depends on the following lemma regarding the commutator of the differential operator L with a smoothing operator.

Let

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon} \right)$$

denote the standard mollifier (η is a compactly supported, non-negative, radially symmetric C^∞-function with unit integral), and let

$$J_\epsilon : L^2(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \qquad J_\epsilon u = \eta_\epsilon * u \tag{8.13}$$

denote the associated smoothing operator.

Lemma 8.11 (Friedrich). *Define* $J_\epsilon : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *by (8.13) and* $L : C_c^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *by (8.3), where* $A^i \in C_c^1(\mathbb{R}^n)$ *and* $C \in C_c(\mathbb{R}^n)$*. Then the commutator*

$$[J_\epsilon, L] = J_\epsilon L - L J_\epsilon, \qquad [J_\epsilon, L] : C_c^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

extends to a bounded linear operator $\overline{[J_\epsilon, L]} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *whose norm is uniformly bounded in* ϵ*. Furthermore, for every* $u \in L^2(\mathbb{R}^n)$

$$\overline{[J_\epsilon, L]}\, u \to 0 \qquad \text{in } L^2(\mathbb{R}^n) \text{ as } \epsilon \to 0^+.$$

Proof. For $u \in C_c^1$, we have

$$\begin{aligned}
[J_\epsilon, L]\, u &= \eta_\epsilon * \left(A^i \partial_i u + Cu\right) - A^i \partial_i \left(\eta_\epsilon * u\right) - C\left(\eta_\epsilon * u\right) \\
&= \eta_\epsilon * \left(A^i \partial_i u\right) - A^i \left(\eta_\epsilon * \partial_i u\right) + \eta_\epsilon * (Cu) - C\left(\eta_\epsilon * u\right).
\end{aligned}$$

By standard properties of mollifiers, if $f \in L^2$ then $\eta_\epsilon * f \to f$ in L^2 as $\epsilon \to 0^+$, so $[J_\epsilon, L]\, u \to 0$ in L^2 when $u \in C_c^1$.

We may write the previous equation as

$$\begin{aligned}
[J_\epsilon, L]\, u(x) = \int \eta_\epsilon(x-y) \Big\{ &\left[A^i(y) - A^i(x)\right] \partial_i u(y) \\
&+ \left[C(y) - C(x)\right] u(y) \Big\}\, dy,
\end{aligned}$$

and an integration by parts gives

$$\begin{aligned}
[J_\epsilon, L]\, u(x) &= \int \partial_i \eta_\epsilon(x-y) \left[A^i(y) - A^i(x)\right] u(y)\, dy \\
&\quad + \int \eta_\epsilon(x-y) \left[C(y) - C(x) - \partial_i A^i(y)\right] u(y)\, dy.
\end{aligned} \tag{8.14}$$

The first term on the right-hand side of (8.14) is bounded uniformly in ϵ because the large factor $\partial_i \eta_\epsilon(x-y)$ is balanced by the factor $A^i(y) - A^i(x)$, which is small on the support of $\eta_\epsilon(x-y)$. To estimate this term, we use the Lipschitz continuity of A^i — with Lipschitz constant K^i, say — to get

$$\begin{aligned}
&\left| \int \partial_i \eta_\epsilon(x-y) \left[A^i(y) - A^i(x)\right] u(y)\, dy \right| \\
&\qquad\qquad \le K^i \int \left|\partial_i \eta_\epsilon(x-y)\right| |x-y| \, |u(y)|\, dy \\
&\qquad\qquad \le K^i \Big[\big(|x \partial_i \eta_\epsilon|\big) * |u| \Big](x).
\end{aligned}$$

Young's inequality implies that

$$\left\| \big(|x\partial_i \eta_\epsilon|\big) * |u| \right\|_{L^2} \le \|x \partial_i \eta_\epsilon\|_{L^1} \|u\|_{L^2},$$

and the L^1-norm

$$E_i = \|x\partial_i \eta_\epsilon\|_{L^1} = \frac{1}{\epsilon^{n+1}} \int |x| \left| \partial_i \eta\left(\frac{x}{\epsilon}\right) \right| \, dx = \int |x| \left|\partial_i \eta(x)\right| \, dx$$

is independent of ϵ. It follows that

$$\left\| \int \partial_i \eta_\epsilon(x-y) \left[A^i(y) - A^i(x)\right] u(y)\, dy \right\|_{L^2} \leq K \|u\|_{L^2}$$

where $K = E_i K^i$.

The second term on the right-hand side of (8.14) is straightforward to estimate:

$$\begin{aligned} \left| \int \eta_\epsilon(x-y) \left[C(y) - C(x) - \partial_i A^i(y)\right] u(y)\, dy \right| & \\ \leq M \int \eta_\epsilon(x-y) \left|u(y)\right| \, dy & \\ \leq M \big(\eta_\epsilon * |u|\big)(x) & \end{aligned}$$

where $M = \sup \left\{2|C| + |\partial_i A^i|\right\}$ is a bound for the coefficient matrices (with $|\cdot|$ denoting the L^2-matrix norm). Young's inequality and the fact that $\|\eta_\epsilon\|_{L_1} = 1$ imply that

$$\begin{aligned} \left\| \int \eta_\epsilon(x-y) \left[C(y) - C(x) - \partial_i A^i(y)\right] u(y)\, dy \right\|_{L^2} & \\ \leq M \left\|\eta_\epsilon * |u|\right\|_{L^2} & \\ \leq M \|u\|_{L^2}. & \end{aligned}$$

Thus, $[J_\epsilon, L]$ is bounded on the dense subset C_c^1 of L^2, so it extends uniquely to a linear operator on L^2 whose norm is bounded by $K + M$ independently of ϵ. Furthermore, since $[J_\epsilon, L]u \to 0$ as $\epsilon \to 0^+$ for all u in a dense subset of L^2, it follows that $\overline{[J_\epsilon, L]}u \to 0$ for all $u \in L^2$. □

Next, we prove the "weak equals strong" theorem.

THEOREM 8.12. *Suppose that the smoothness assumptions in Definition 8.1 are satisfied, $B = \nu_i A^i$ is nonsingular on $\partial\Omega$, and $f \in L^2(\Omega)$. Then a function $u \in L^2(\Omega)$ is a weak solution of (8.1) if and only if it is a strong solution.*

PROOF. Suppose u is a strong solution of (8.1), meaning that there is a sequence (u_n) of smooth solutions such that $u_n \to u$ and $Lu_n \to f$ in $L^2(\Omega)$ as $n \to \infty$. These solutions satisfy $(u_n, L^* v) = (Lu_n, v)$ for all $v \in D^*$, and taking the limit of this equation as $n \to \infty$, we get that $(u, L^* v) = (f, v)$ for all $v \in D^*$. This means that u is a weak solution.

To prove that a weak solution is a strong solution, we use a partition of unity to localize the problem and mollifiers to smooth the weak solution. In the interior of the domain, we use a standard mollifier. On the boundary, we make a change of coordinates to "flatten" the boundary and mollify only in the tangential directions to preserve the boundary condition. The smoothness of the mollified solution in the normal direction then follows from the PDE, since we can express the normal derivative of a solution in terms of the tangential derivatives if the boundary is non-characteristic.

In more detail, suppose that $u \in L^2(\Omega)$ is a weak solution of (8.1), meaning that $(u, L^* v) = (f, v)$ for all $v \in D^*$, where $(\cdot, \cdot)$ denotes the standard inner product on $L^2(\Omega)$ and D^* is defined in (8.10).

Let $\{U_j\}$ be a finite open cover of $\overline{\Omega}$ by interior or boundary coordinate patches U_j. An interior patch is compactly contained in Ω and diffeomorphic to a ball

$\{|x| < 1\}$; a boundary patch intersects Ω in a region that is diffeomorphic to a half-ball $\{x_1 > 0,\ |x| < 1\}$. Introduce a subordinate partition of unity $\{\phi_j\}$ with $\operatorname{supp}\phi_j \subset U_j$ and $\sum_j \phi_j = 1$ on $\overline{\Omega}$, and let

$$u = \sum_j u_j, \qquad u_j = \phi_j u.$$

We claim that $u \in L^2(\Omega)$ is a weak solution of $Lu = f$, with the boundary condition $B_- u = 0$, if and only if each $u_j \in L^2(\Omega)$ is a weak solution of

$$Lu_j = \phi_j f + [\partial_i \phi_j] A^i u, \tag{8.15}$$

with the same boundary condition. The right-hand side of (8.15) depends on u, but it belongs to $L^2(\Omega)$ since it involves no derivatives of u.

To verify this claim, suppose that $Lu = f$. Then, by use of the equations $(u, \phi v) = (\phi u, v)$ and $(u, L^* v) = (f, u)$, we get for all $v \in D^*$ that

$$\begin{aligned}(u_j, L^* v) &= (u, \phi_j L^* v)\\ &= (u, L^*[\phi_j v]) + \big(u, [\partial_i \phi_j] A^i v\big)\\ &= (f, \phi_j v) + \big(u, [\partial_i \phi_j] A^i v\big)\\ &= \big(\phi_j f + [\partial_i \phi_j] A^i u, v\big),\end{aligned} \tag{8.16}$$

which shows that u_j is a weak solution of (8.15). Conversely, suppose that u_j a weak solution of (8.15). Then by summing (8.16) over j and using the equation $\sum_j [\partial_i \phi_j] = 0$, we find that $u = \sum_j u_j$ is a weak solution of $Lu = f$. Thus, to prove that a weak solution u is a strong solution it suffices to prove that each u_j is a strong solution. We may therefore assume without loss of generality that u is supported in an interior or boundary patch.

First, suppose that u is supported in an interior patch. Since $u \in L^2(\Omega)$ is compactly supported in Ω, we may extend u by zero on Ω^c and extend other functions to compactly supported functions on $\mathbb{R}^n$. Then $u_\epsilon = J_\epsilon u \in C_c^\infty$ is well-defined and, by standard properties of mollifiers, $u_\epsilon \to u$ in L^2 as $\epsilon \to 0^+$. We will show that $Lu_\epsilon \to Lu$ in L^2, which proves that u is a strong solution.

Using the self-adjointness of J_ϵ, we have for all $v \in D^*$ that

$$\begin{aligned}(u_\epsilon, L^* v) &= (u, J_\epsilon L^* v)\\ &= (u, L^* J_\epsilon v) + (u, [J_\epsilon, L^*] v)\\ &= (f, J_\epsilon v) + (u, [J_\epsilon, L^*] v)\\ &= (J_\epsilon f, v) + (u, [J_\epsilon, L^*] v).\end{aligned}$$

Lemma 8.11, applied to L^*, implies that $[J_\epsilon, L^*]$ is bounded on L^2. Moreover, a density argument shows that its Hilbert-space adjoint is

$$[J_\epsilon, L^*]^* = -\overline{[J_\epsilon, L]}.$$

Thus,

$$(u_\epsilon, L^* v) = \Big(J_\epsilon f - \overline{[J_\epsilon, L]} u, v\Big) \qquad \text{for all } v \in D^*,$$

which means that u_ϵ is a weak solution of

$$Lu_\epsilon = f_\epsilon, \qquad f_\epsilon = J_\epsilon f - \overline{[J_\epsilon, L]} u.$$

Since u_ϵ is smooth, it is a classical solution that satisfies the boundary condition $B_- u_\epsilon = 0$ pointwise.. Lemma 8.11 and the properties of mollifiers imply that $f_\epsilon \to f$ in L^2 as $\epsilon \to 0^+$, which proves that u is a strong solution.

Second, suppose that u is supported in a boundary patch $\overline{\Omega} \cap U_j$. In this case, we obtain a smooth approximation by mollifying u in the tangential directions. The PDE then implies that u is smooth in the normal direction.

By making a C^2-change of the independent variable, we may assume without loss of generality that Ω is a half-space

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x^1 > 0\},$$

and u is compactly supported in $\overline{\mathbb{R}}^n_+$. We write

$$x = (x^1, x'), \qquad x' = (x^2, \dots, x^n) \in \mathbb{R}^{n-1}.$$

Since we assume that the boundary is non-characteristic, A^1 is nonsingular on $x^1 = 0$, in which case it is non-singular in a neighborhood of the boundary by continuity. Restricting the support of u appropriately, we may assume that A^1 is nonsingular everywhere, and multiplication of the PDE by the inverse matrix puts the equation $Lu = f$ in the form

$$\partial_1 u + L'u = f, \qquad L' = A^{i'} \partial_{i'} + C \qquad \text{in } x^1 > 0, \tag{8.17}$$

where the sum is taken over $2 \le i' \le n$, and the matrices $A^{i'}(x^1, x')$ need not be symmetric. The weak form of the equation transforms correspondingly under a smooth change of independent variable.

We may regard $u \in L^2(\mathbb{R}^n_+)$ equivalently as a vector-valued function of the normal variable $u \in L^2\left(\mathbb{R}_+; L^2\right)$ where $u : x^1 \mapsto u(x^1, \cdot)$, and we abbreviate the range space $L^2(\mathbb{R}^{n-1})$ of functions of the tangential variable x' to L^2. If $(\cdot, \cdot)'$ denotes the L^2-inner product with respect to $x' \in \mathbb{R}^{n-1}$, then the inner product on this space is the same as the $L^2(\mathbb{R}^n)$-inner product:

$$(u, v)_{L^2(\mathbb{R}_+; L^2)} = \int_{\mathbb{R}_+} (u, v)' \, dx^1 = (u, v).$$

We denote other spaces similarly. For example, $L^2\left(\mathbb{R}_+; H^1\right)$ consists of functions with square-integrable tangential derivatives, with inner product

$$(u, v)_{L^2(\mathbb{R}_+; H^1)} = \int_{\mathbb{R}_+} \left\{ (u, v)' + \sum_{i'=2}^{n} (\partial_{i'} u, \partial_{i'} v)' \right\} dx^1;$$

and $H^1\left(\mathbb{R}_+; L^2\right)$ consists of functions with square-integrable normal derivatives, with inner product

$$(u, v)_{H^1(\mathbb{R}_+; L^2)} = \int_{\mathbb{R}_+} \left\{ (u, v)' + (\partial_1 u, \partial_1 v)' \right\} dx^1.$$

In particular, $H^1(\mathbb{R}^n_+) = L^2\left(\mathbb{R}_+; H^1\right) \cap H^1\left(\mathbb{R}_+; L^2\right)$.

Let η'_ϵ be the standard mollifier with respect to $x' \in \mathbb{R}^{n-1}$, and define the associated tangential smoothing operator $J'_\epsilon : u \mapsto u_\epsilon$ by

$$u_\epsilon(x^1, x') = \int_{\mathbb{R}^{n-1}} \eta'_\epsilon(x' - y') u(x^1, y') \, dy'.$$

If $u \in L^2(\mathbb{R}^n_+)$, then $u_\epsilon \in L^2\left(\mathbb{R}_+; H^1\right)$. Fubini's theorem and standard properties of mollifiers imply that $u_\epsilon \to u$ in $L^2(\mathbb{R}^n_+)$ as $\epsilon \to 0^+$.

Mollifying the weak form of (8.17) in the tangential directions and using the fact that J'_ϵ commutes with ∂_1, we get — as in the interior case — that

$$\begin{aligned}(u_\epsilon, L^* v) &= \left(u_\epsilon, \left\{-\partial_1 + L'^*\right\} v\right)\\ &= \left(u, \left\{-\partial_1 + L'^*\right\} J'_\epsilon v\right) + \left(u, \left[J'_\epsilon, L'^*\right] v\right)\\ &= \left(J'_\epsilon f - \overline{[J'_\epsilon, L']}u, v\right),\end{aligned}$$

meaning that u_ϵ is a weak solution of

$$\partial_1 u_\epsilon + L' u_\epsilon = f_\epsilon, \qquad f_\epsilon = J'_\epsilon f - \overline{[J'_\epsilon, L']}u. \tag{8.18}$$

Lemma 8.11 applied to the tangential commutator implies that

$$\overline{[J'_\epsilon, L']}u \in L^2(\mathbb{R}_+; L^2)$$

and $f_\epsilon \to f$ in $L^2(\mathbb{R}_+; L^2)$. Moreover, (8.18) shows that $u_\epsilon \in H^1(\mathbb{R}_+; L^2)$. Thus, we have constructed $u_\epsilon \in H^1(\mathbb{R}^n_+)$ such that

$$u_\epsilon \to u, \quad Lu_\epsilon \to Lu \qquad \text{in } L^2(\mathbb{R}^n_+) \text{ as } \epsilon \to 0^+. \tag{8.19}$$

In view of (8.19), we just need to show that weak H^1-solutions are strong solutions.[1] By making a linear transformation of u, we can transform the boundary condition $B_- u = 0$ into

$$u_1 = u_2 = \cdots = u_r = 0,$$

where r is the dimension of $\ker B_-$. We decompose $u = u^+ + u^-$ where

$$u^+ = (u_1, \dots, u_r, 0, \dots, 0)^T, \qquad u^- = (0, \dots, 0, u_{r+1}, \dots, u_n)^T,$$

in which case the boundary condition is $u^+ = 0$ on $x^1 = 0$, with u^- arbitrary.

If $u \in H^1(\mathbb{R}^n_+)$ is a weak solution of $Lu = f$, then $\gamma u^+ = 0$, where γ is the trace map in (8.11). This condition implies that [**9**]

$$u^+ \in H^1_0(\mathbb{R}^n_+), \qquad H^1_0(\mathbb{R}^n_+) = \overline{C_c^\infty(\mathbb{R}^n_+)}.$$

Consequently, there exist $u_\epsilon^+ \in C_c^1(\mathbb{R}^n_+)$ such that $u_\epsilon^+ \to u^+$ in $H^1(\mathbb{R}^n_+)$ as $n \to \infty$. Since u_ϵ^+ has compact support in $\mathbb{R}^n_+$, it satisfies the boundary condition pointwise. Furthermore, by density, there exist $u_\epsilon^- \in C_c^1(\overline{\mathbb{R}}^n_+)$ such that $u_\epsilon^- \to u^-$ in $H^1(\mathbb{R}^n_+)$. Let $u_\epsilon = u_\epsilon^+ + u_\epsilon^-$. Then $u_\epsilon \in C_c^1(\overline{\mathbb{R}}^n_+)$, $B_- u_\epsilon = 0$, and $u_\epsilon \to u$ in $H^1(\mathbb{R}^n_+)$. Since $L : H^1(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n_+)$ is bounded, $u_\epsilon \to u$, $Lu_\epsilon \to Lu$ in $L^2(\mathbb{R}^n_+)$, which proves that u is a strong solution. □

If the boundary is not smooth, or the boundary matrix B is singular and the dimension of its null-space changes, then difficulties may arise with the tangential mollification near the boundary; in that case weak solutions might not be strong solutions e.g. see [**30**].

Note that Theorem 8.12 is based entirely on mollification and does not depend on any positivity or symmetry conditions

Corollary 8.13. *Let $f \in L^2(\Omega)$. If the smoothness conditions in Definition 8.1 and the positivity conditions in Definition 8.4, Definition 8.5 are satisfied, then a weak solution $u \in L^2(\Omega)$ of (8.1) is unique and $c\|u\| \le \|f\|$.*

[1] If we had defined strong solutions equivalently as the limit of H^1-solutions instead of C^1-solutions, we wouldn't need this step.

PROOF. Let $u \in L^2(\Omega)$ be a weak solution of (8.1). By Theorem 8.12, there is a sequence (u_n) of smooth solutions $u_n \in C^1(\overline{\Omega})$ of (8.1) with $Lu_n = f_n$ such that $u_n \to u$ and $f_n \to f$ in L^2. Theorem 8.6 implies that $c\|u_n\| \leq \|f_n\|$ and, taking the limit of this inequality as $n \to \infty$, we get $c\|u\| \leq \|f\|$. In particular, $f = 0$ implies that $u = 0$, so a weak solution is unique. □

A further issue is the regularity of weak solutions, which follows from energy estimates for their derivatives. As shown in Rauch [**33**] and the references cited there, if the boundary is non-characteristic, then the solution is as regular as the data allows: If A^i and $\partial\Omega$ are C^{k+1}, C is C^k, and $f \in H^k(\Omega)$, then $u \in H^k(\Omega)$.