

**Notes on
Partial Differential Equations**

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ABSTRACT. These are notes from a two-quarter class on PDEs that are heavily based on the book *Partial Differential Equations* by L. C. Evans, together with other sources that are mostly listed in the Bibliography. The notes cover roughly Chapter 2 and Chapters 5–7 in Evans. There is no claim to any originality in the notes, but I hope — for some readers at least — they will provide a useful supplement.

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