Analysis Preliminary Exam Workshop: Metric and Banach Spaces

1. Metric spaces

A metric space is complete if every Cauchy sequence converges. A metric space is compact (every open cover has a finite subcover) if and only if it is sequentially compact (every bounded sequence has a convergent subsequence).

THEOREM 1 (Compactness). *A metric space is compact if and only if it is complete and totally bounded (i.e. for every $\epsilon > 0$ there is a finite cover of the space by balls of radius ϵ).*

THEOREM 2 (Contraction mapping). *If (X, d) is a complete metric space and $f : X \to X$ is a strict contraction, meaning that there is a constant $0 \leq k < 1$ such that*

$$d\left(f(x)f(y)\right) < kd(x, y) \qquad \textit{for all } x, y \in X,$$

then f has a unique fixed point $x \in X$ such that $f(x) = x$.

2. Banach spaces

A Banach space $(X, \|\cdot\|)$ is a complete normed linear space (real or complex).

Examples: $\mathbb{R}^n$ or $\mathbb{C}^n$ with the p-norm; the space $C(K)$ of continuous functions on a compact set K with the maximum (or sup) norm; the L^p and ℓ^p spaces.

A linear map $A : X \to Y$ between Banach spaces is bounded (equivalent to continuous for linear maps) if and only if its operator norm

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

is finite.

THEOREM 3. *The space $\mathcal{B}(X, Y)$ of bounded linear operators between Banach spaces X, Y with the operator norm is a Banach space. The space $\mathcal{B}(X)$ of bounded linear operators on X is a Banach algebra with $\|AB\| \leq \|A\| \|B\|$ for all $A, B \in \mathcal{B}(X)$*

3. Dual spaces

The dual space X' (or X^*) of a Banach space X is the Banach space of all bounded linear functionals $F : X \to \mathbb{C}$ (or $\mathbb{R}$ if X is a real Banach space) with the operator norm

$$\|F\| = \sup_{x \neq 0} \frac{|F(x)|}{\|x\|}.$$

THEOREM 4 (Hahn-Banach). *Suppose that Y is a linear subspace (not necessarily closed) of a normed vector space X and $f : Y \to \mathbb{C}$ (or $\mathbb{R}$) is a bounded linear functional on Y. There exists an extension F of f to a bounded linear functional $F : X \to \mathbb{C}$ (or $\mathbb{R}$) with $\|F\| = \|f\|$.*

One consequence is that for every $x \in X$

$$\|x\| = \sup \{|F(x)| : F \in X' \text{ and } \|F| \ = 1\}.$$

We identify $x \in X$ with $x \in X''$ by $x : F \mapsto F(x)$ i.e. we think of x as acting on F instead of F acting on x. A Banach space is reflexive if $X'' = X$.

If X is a Banach space with dual space X' and (x_n) is a sequence in X, then (x_n) converges weakly to $x \in X$, written $x_n \rightharpoonup x$, if

$$F(x_n) \to F(x) \qquad \text{for every } F \in X'.$$

If $X = Y'$ is the dual of a Banach space Y, then (x_n) in X converges weak-$*$ to $x \in X$, written $x_n \overset{*}{\rightharpoonup} x$, if

$$x_n(F) \to x(F) \qquad \text{for every } F \in Y.$$

If X is reflexive, then weak and weak-$*$ convergence are equivalent.

THEOREM 5 (Banach-Alaoglu). *The closed unit ball in a Banach space $X = Y'$ is weak-$*$ compact.*

4. Spaces of continuous functions

Let K be a compact metric space (e.g. a closed bounded subset of $\mathbb{R}^n$) and $C(K)$ the space of continuous functions $f : K \to \mathbb{C}$ with the sup-norm

$$\|f\|_{L^\infty} = \sup_K |f|.$$

(The sup is attained for continuous f on a compact set K.)

Theorem 6. *If K is a compact metric space, then $(C(K), \|\cdot\|_{L^\infty})$ is a Banach space.*

Theorem 7 (Arzelà-Ascoli). *Let (K, d) be a compact metric space. A subset E of $C(K)$ is compact if and only if it is closed, bounded, and uniformly equicontinuous i.e. for every $\epsilon > 0$ there exists $\delta > 0$ (*independent *of $f \in E$) such that*

$$|f(x) - f(y)| < \epsilon \qquad \text{for every } f \in E \text{ whenever } d(x, y) < \delta.$$

Theorem 8 (Weierstrass). *The polynomials are dense in $C([a, b])$.*

A generalization of this theorem is the Stone-Weierstrass theorem.

5. ℓ^p spaces

Let $\ell^p(\mathbb{N})$ denote the space of all sequences $(x_n)_{n=1}^\infty$ of real or complex numbers such that

$$\|x\|_{\ell^p} = \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} < \infty$$

if $1 \le p < \infty$, or

$$\|x\|_{\ell^\infty} = \sup_{n \in \mathbb{N}} |x_n| < \infty$$

if $p = \infty$.

Theorem 9. *If $1 \le p \le \infty$, the space $(\ell^p(\mathbb{N}), \|\cdot\|_{\ell^p})$ is a Banach space.*

6. L^p spaces

If Ω is a a Lebesgue measurable subset of $\mathbb{R}^n$, such as an open set, then $L^p(\Omega)$ is the space of measurable functions $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$), identified up to pointwise almost everywhere equality, with norm

$$\|f\|_{L^p} = \left(\int_\Omega |f|^p \, dx \right)^{1/p}$$

if $1 \leq p < \infty$, or

$$\|f\|_{L^\infty} = \sup_\Omega |f|$$

if $p = \infty$, where sup is the essential supremum.

Theorem 10. *If $1 \leq p \leq \infty$, the space $(L^p(\Omega), \|\cdot\|_{L^p})$ is a Banach space.*

Theorem 11. *Every sequence of L^p-functions that converges in $L^p(\Omega)$ has a subsequence that converges pointwise almost everywhere.*

Theorem 12 (Density). *If $\Omega \subset \mathbb{R}^n$ is an open set and $1 \leq p < \infty$, then the space $C_c(\Omega)$ of continuous functions on Ω with compact support in Ω is dense in $L^p(\Omega)$.*

Mollification shows that $C_c^\infty(\Omega)$ is also dense in $L^p(\Omega)$. This theorem is false for $p = \infty$ (since the L^∞-limit of continuous functions is continuous).

Theorem 13 (Dual of L^p). *Suppose that Ω is a measurable subset of $\mathbb{R}^n$ and $1 \leq p < \infty$. Then the dual space of $L^p(\Omega)$ is isomorphic to $L^{p'}(\Omega)$ where p' is the Hölder conjugate of p, such that*

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

and $F \in (L^p)' \mapsto f \in L^{p'}$ by

$$F(g) = \int_\Omega fg \, dx \qquad \text{for all } g \in L^p.$$

This theorem fails for $p = \infty$: L^∞ is the dual space of L^1, but the dual space of L^∞ is typically much larger than L^1. Thus, L^p is reflexive for $1 < p < \infty$, but is typically non-reflexive for $p = 1, \infty$.

7. Fourier transform

If $f : \mathbb{R}^n \to \mathbb{C}$ is a measurable function, the Fourier transform $\hat{f} = \mathcal{F}f$, where $\hat{f} : \mathbb{R}^n \to \mathbb{C}$, and the inverse Fourier transform $f = \mathcal{F}^{-1}f$ are defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int f(x) e^{-ix\xi}\, dx, \qquad f(x) = \frac{1}{(2\pi)^{n/2}} \int \hat{f}(\xi) e^{ix\xi}\, d\xi$$

provided these integrals exist.

We consider the Fourier transform on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, $L^1(\mathbb{R}^n)$, and $L^2(\mathbb{R}^n)$. Later we will consider the Fourier transform on the space $\mathcal{S}(\mathbb{R}^n)'$ of tempered distributions.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of smooth, rapidly decreasing functions. That is $f \in \mathcal{S}(\mathbb{R}^n)$ if $f \in C^\infty(\mathbb{R}^n)$ and

$$x^\alpha \partial^\beta f \to 0 \qquad \text{as } |x| \to \infty$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. For example, $\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 14 (Fourier transform on Schwartz functions). *The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}_n)$ is one-to-one and onto, with inverse $\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}_n)$ as defined above. Moreover*

$$\mathcal{F}\left(x^\alpha f\right) = i^{|\alpha|} \partial_\xi^\alpha \hat{f}, \qquad \mathcal{F}\left(\partial_x^\beta f\right) = i^{|\beta|} \xi^\beta \hat{f}.$$

Thus, the Fourier transform exchanges smoothness and decay at infinity.

Theorem 15 (Riemann-Lebesgue). *The Fourier transform maps $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ and*

$$\|\hat{f}\|_{L^\infty} \le \frac{1}{(2\pi)^{n/2}} \|f\|_{L^1}.$$

Explicitly, if $f \in L^1$ then $\hat{f}$ is continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Theorem 16 (Plancherel). *The Fourier transform on $L^1 \cap L^2$ extends uniquely to a unitary map $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. In particular,*

$$\|f\|_L^2 = \|\hat{f}\|_{L^2}.$$

Interpolation theory then implies the following result.

Theorem 17 (Hausdorff-Young). *If* $1 \le p \le 2$ *and* $2 \le p' \le \infty$ *is its Hölder conjugate, then* $\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ *is a bounded linear map.*

Note that $\mathcal{F}$ is not onto unless $p = 2$, and this result is false for $3 < p \le \infty$.

Theorem 18 (Convolution). *If* $f, g \in L^1(\mathbb{R}^n)$*, then* $f * g \in L^1(\mathbb{R}^n)$ *and*

$$\widehat{f * g} = (2\pi)^{n/2} \hat{f} \hat{g}$$

More generally, this result applies if $f \in L^p$*,* $g \in L^q$ *and* $1 \le p, q, r \le 2$ *where*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

in which case $f * g \in L^r$ *and* $\mathcal{F}(f * g) \in L^{r'}$*.*