Analysis Preliminary Exam Workshop: Basic Theorems for Integration

1. Notation

If $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable subset of $\mathbb{R}^n$ and $1 \leq p \leq \infty$, then $L^p(\Omega)$ is the space of all Lebesgue measurable functions (equivalent up to almost everywhere equality) such that

$$\|f\|_{L^p} = \left(\int_\Omega |f|^p \, dx \right)^{1/p} < \infty$$

if $1 \leq p < \infty$, or

$$\|f\|_{L^\infty} = \sup\{|f(x)| : x \in \Omega\} < \infty$$

if $p = \infty$ (where sup denotes the essential supremum). We mostly consider functions defined on $\mathbb{R}^n$; results for general domains Ω follow by applying these results to $f\chi_\Omega$ where χ_Ω is the characteristic function of Ω. We consider real-valued functions for definiteness, but all the results (except for the Monotone Convergence Theorem and Fatou's Lemma, which depend on the ordering properties of $\mathbb{R}$) extend to complex-valued functions in the obvious way.

2. Exchanging the order of limits and integration

THEOREM 1 (Monotone convergence). *Suppose that (f_n) is an increasing sequence of non-negative measurable functions $f_n : \mathbb{R}^n \to [0, \infty]$ and $f = \lim_{n\to\infty} f_n$. Then*

$$\int f \, dx = \lim_{n\to\infty} \int f_n \, dx.$$

THEOREM 2 (Fatou's lemma). *Suppose that (f_n) is a sequence of non-negative, measurable functions $f_n : \mathbb{R}^n \to [0, \infty]$. Then*

$$\int \liminf_{n\to\infty} f_n \, dx \leq \liminf_{n\to\infty} \int f_n \, dx.$$

This result says that integrals can only “lose mass” in the limit.

THEOREM 3 (Dominated convergence). *Suppose that (f_n) is a sequence of integrable functions $f_n \in L^1(\mathbb{R}^n)$ and $f = \lim_{n\to\infty} f_n$. If there exists $g \in L^1(\mathbb{R}^n)$ such that*

$$|f_n| \leq g \qquad \text{for all } n \in \mathbb{N},$$

then $f \in L^1(\mathbb{R}^n)$ and

$$\int f\, dx = \lim_{n\to\infty} \int f_n\, dx.$$

3. Continuity of integrals and differentiation under the integral

THEOREM 4. *Suppose that $f : \mathbb{R}^n \times [a,b] \to \mathbb{R}$ is such that*

$$f(\cdot, t) : \mathbb{R}^n \to \mathbb{R} \in L^1(\mathbb{R}^n) \qquad \text{for each } a \leq t \leq b.$$

Define $F : [a,b] \to \mathbb{R}$ by

$$F(t) = \int f(x,t)\, dx.$$

(a) If $f(x, \cdot) : [a,b] \to \mathbb{R}$ is continuous at $a \leq t_0 \leq b$ for every $x \in \mathbb{R}^n$ and there exists a function $g \in L^1(\mathbb{R}^n)$ such that

$$|f(x,t)| \leq g(x) \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [a,b],$$

then F is continuous at t_0.
(b) If the partial derivative

$$\frac{\partial f}{\partial t}(x,t)$$

exists for all $(x,t) \in \mathbb{R}^n \times [a,b]$ and there is a function $g \in L^1(\mathbb{R}^n)$ such that

$$\left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x) \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [a,b],$$

then F is differentiable on $[a,b]$ and

$$\frac{dF}{dt}(t) = \int \frac{\partial f}{\partial t}(x,t)\, dx.$$

4. Change of variables in integrals

Let $U, V \subset \mathbb{R}^n$ be open sets in $\mathbb{R}^n$. A map $\phi : U \to V$ is a C^1-diffeomorphism of U onto V if it is one-to-one and onto, ϕ is continuously differentiable in U, and ϕ^{-1} is continuously differentiable in V.

THEOREM 5. *Suppose that $f : V \to \mathbb{R}$ is measurable function defined on an open set $V \subset \mathbb{R}^n$ and $\phi : U \to V$ is a C^1-diffeomorphism of U onto V. Let*

$$J = |\det D\phi|$$

be the Jacobian of ϕ. Then $f \in L^1(V)$ if and only if $(f \circ \phi)J \in L^1(U)$, in which case

$$\int_V f(y)\, dy = \int_U f(\phi(x))\, J(x)\, dx.$$

5. Evaluation of double integrals and exchange in the order of integration

THEOREM 6 (Fubini). *Suppose that $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function. Then*

$$\int |f(x, y)|\, dxdy$$

is finite if and only if either one of

$$\int \left(\int |f(x, y)|\, dx \right) dy, \qquad \int \left(\int |f(x, y)|\, dy \right) dx$$

is finite. In that case,

$$\int f(x, y)\, dxdy = \int \left(\int f(x, y)\, dx \right) dy = \int \left(\int f(x, y)\, dy \right) dx.$$

6. Inequalities

THEOREM 7 (Hölder's inequality). *Suppose that* $1 \le p, q \le \infty$ *and*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, *then*

$$\left| \int fg \, dx \right| \le \left(\int |f|^p \, dx \right)^{1/p} \left(\int |g|^q \, dx \right)^{1/q}.$$

Special cases:
(a) $p = q = 2$ is the Cauchy-Schwartz inequality

$$\left| \int fg \, dx \right| \le \left(\int f^2 \, dx \right)^{1/2} \left(\int g^2 \, dx \right)^{1/2};$$

(b) $p = 1$, $q = \infty$ is

$$\left| \int fg \, dx \right| \le \sup |g| \int |f| \, dx.$$

If $f, g : \mathbb{R}^n \to \mathbb{R}$ are measurable functions, then the convolution

$$f * g : \mathbb{R}^n \to \mathbb{R}$$

of f and g is the function

$$(f * g)(x) = \int f(x - y) g(y) \, dy,$$

provided this integral exists.

THEOREM 8 (Young's inequality). *Suppose that* $1 \le p, q, r \le \infty$ *and*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

If $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$, *then* $f * g \in L^r(\mathbb{R}^n)$ *and*

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}.$$

Special cases:
(a) $p = q = 2$, $r = 1$ is

$$\sup |f * g| \leq \|f\|_{L^2} \|g\|_{L^2};$$

(b) $q = 1$, $p = r$ is

$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}.$$

From (b), convolution with an L^1 function is a bounded linear map on L^p.

THEOREM 9 (Jensen's inequality). *Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function and $f \in L^1(\Omega)$ where $\Omega \subset \mathbb{R}$ has finite Lebesgue measure $|\Omega| > 0$. Then*

$$\phi\left(\frac{1}{|\Omega|} \int_\Omega f(x)\, dx\right) \leq \frac{1}{|\Omega|} \int_\Omega \phi\left(f(x)\right)\, dx.$$

This says that the value of a convex function at an average is less than or equal to the average of its values.