Sample Prelim Problems
Metric and Banach Spaces
Fall 2012

1. Let (X, d) be a compact metric space, and $f : X \to X$ a weak contraction, meaning that $d(f(x), f(y)) < d(x, y)$ if $x \neq y$. Show that f has a fixed point. Hint: Consider $d(x_n, f(x_n))$.

2. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be continuous and define a map

$$K : C([0,1]) \to C([0,1]), \qquad (Kf)(x) = \int_0^1 k(x,y) f(y)\, dy.$$

Prove that K is compact.

3. Suppose $f \in C^k([0,1])$ is k-times continuously differentiable. Prove that for every $\epsilon > 0$ there is a polynomial P such that for all $x \in [0,1]$

$$|f(x) - P(x)| < \epsilon, \quad |f'(x) - P'(x)| < \epsilon, \ldots, \left|f^{(k)}(x) - P^{(k)}(x)\right| < \epsilon.$$

4. (Fall, 2007) Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be Banach spaces with X compactly embedded in Y, and Y continuously embedded in Z (meaning that $X \subset Y \subset Z$, bounded sets in $(X, \|\cdot\|_X)$ are precompact in $(Y, \|\cdot\|_Y)$, and there is a constant M such that $\|x\|_Z \leq M\|x\|_Y$ for every $x \in Y$). Prove that for every $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that

$$\|x\|_Y \leq \epsilon \|x\|_X + C(\epsilon)\|x\|_Z \qquad \text{for every } x \in X.$$

5. Suppose that $1 \leq p < \infty$ and $h \in \mathbb{R}^n$. For $f \in L^p(\mathbb{R}^n)$ defined $\tau_h f \in L^p(\mathbb{R}^n)$ by $(\tau_h f)(x) = f(x+h)$. Show that $\tau_h f \to f$ in $L^p(\mathbb{R}^n)$ as $h \to 0$. Give a counter-example to this result if $p = \infty$.

6. (Spring, 2011) Let $\Omega = (0,1)$ be the open unit interval in $\mathbb{R}$, and consider the sequence of functions $f_n(x) = ne^{-nx}$. Prove that the sequence (f_n) does not converge weakly in $L^1(0,1)$. (Hint: Prove by contradiction.)

7. (a) If $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{C}$ is measurable and $1 \leq p < \infty$, prove that

$$\left[\int \left(\int |f(x,y)|\, dy\right)^p dx\right]^{1/p} \leq \int \left[\int |f(x,y)|^p\, dx\right]^{1/p} dy.$$

(b) If $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, deduce that $f * g \in L^p(\mathbb{R}^n)$ and prove Young's inequality

$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}.$$

Hint: For (a), note that by duality

$$\left[\int \left(\int |f(x,y)|\, dy \right)^p dx \right]^{1/p} = \sup_{\|g\|_{L^{p'}}=1} \int \left(\int |f(x,y)|\, dy \right) g(x)\, dx$$

where $1 < p' \leq \infty$ is the Hölder conjugate of p and the supremum is taken over all $g \in L^{p'}(\mathbb{R}^n)$ such that $\|g\|_{l^{p'}} = 1$.

8. (Winter, 2009) Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of smooth, rapidly decreasing functions. Define an operator $H : \mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\widehat{(Hf)}(\xi) = i \operatorname{sgn}(\xi) \hat{f}(\xi) = \begin{cases} i\hat{f}(\xi) & \text{if } \xi > 0 \\ -i\hat{f}(\xi) & \text{if } \xi < 0 \end{cases}$$

where $\hat{f}$ denotes the Fourier transform of f.
(a) Why is $Hf \in L^2(\mathbb{R})$ for every $f \in \mathcal{S}(\mathbb{R})$?
(b) If $f \in \mathcal{S}(\mathbb{R})$ and $Hf \in L^1(\mathbb{R})$ show that

$$\int_{\mathbb{R}} f(x)\, dx = 0.$$

Hint: Riemann-Lebesgue Lemma.

9. (a) For $f \in L^1(\mathbb{R})$ and $R > 0$, let

$$(S_R f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{f}(\xi) e^{ix\xi}\, d\xi$$

where $\hat{f}$ is the Fourier transform of f, defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\xi}\, dx.$$

Show that

$$S_R f = K_R * f$$

where

$$K_R(x) = \frac{\sin Rx}{\pi x}$$

Show that the result also holds if $f \in L^2(\mathbb{R})$.
(b) If $f \in L^2$, show that $S_R f \to f$ in L^2 as $R \to \infty$.