SAMPLE PRELIM PROBLEMS
HILBERT SPACES
Fall 2012

1. (Fall, 2007) Let $\mathcal{H}$ be the weighted L^2 space

$$\mathcal{H} = \left\{ f : \mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} e^{-|x|} |f(x)|^2 \, dx < \infty \right\}$$

with inner product

$$(f, g) = \int_{\mathbb{R}} e^{-|x|} \overline{f(x)} g(x) \, dx$$

Let $T : \mathcal{H} \to \mathcal{H}$ be the translation operator

$$(Tf)(x) = f(x+1).$$

Compute the adjoint T^* and operator norm $\|T\|$.

2. (Fall, 2011) Let $\mathcal{H}$ be a complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear transformations (operators) of $\mathcal{H}$ considered with the operator norm.

(a) What does it mean for $A \in \mathcal{B}(\mathcal{H})$ to be compact? Give a definition of compactness of an operator A in terms of properties of the image of bounded sets, e.g., the set $\{Ax \mid x \in \mathcal{H}, \|x\| \le 1\}$.

(b) Suppose that $\mathcal{H}$ is separable and let $\{e_n\}_{n \ge 0}$ be an orthonormal basis of $\mathcal{H}$. For $n \ge 0$, let P_n denote the orthogonal projection onto the subspace spanned by $e_0, \ldots, e_n$. Prove that $A \in \mathcal{B}(\mathcal{H})$ is compact iff the sequence $(P_n A)_{n \ge 0}$ converges to A in norm.

3. (Spring, 2010) Suppose that $h : [0,1]^2 \to [0,1]^2$ is a continuously differentiable function from the square to the square with continuously differentiable inverse h^{-1}. Define an operator T on the Hilbert space $L^2([0,1]^2)$ by the formula $T(f) = f \circ h$. Prove that T is a well-defined bounded operator on this Hilbert space.

4. An operator $A \in \mathcal{B}(\mathcal{H})$ is normal if it commutes with its adjoint. Define $V : L^2(0,1) \to L^2(0,1)$ by

$$(Vf)(x) = \int_0^x f(t) \, dt$$

and $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$$

Are either of V, S normal?

5. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$.
(a) If $(x_k)_{k=1}^\infty$ is a sequence in $\mathcal{H}$, show that $x_k \rightharpoonup 0$ as $k \to \infty$ if and only if

$$(e_n, x_k) \to 0 \qquad \text{as } k \to \infty \text{ for every } n \in \mathbb{N}$$

and $\{\|x_k\| : k \in \mathbb{N}\}$ is bounded. (You can assume the Theorem that every weakly convergent sequence is bounded.)
(b) Give an example of a sequence (x_k) in $\mathcal{H}$ such that $(e_n, x_k) \to 0$ as $k \to \infty$ for every $n \in \mathbb{N}$ but (x_k) does not converge weakly to 0.

6. If $T \in \mathcal{B}(\mathcal{H})$ is a bounded linear operator on a Hilbert space $\mathcal{H}$, prove that T is compact if and only if it maps weakly convergent sequences to strongly convergent sequences.

7. (a) If $A, B \in \mathcal{B}(\mathcal{H})$ are bounded linear operators on a Hilbert space $\mathcal{H}$, prove that

$$\|AB\| \leq \|A\| \, \|B\|.$$

(b) If $A \in \mathcal{B}(\mathcal{H})$, prove that $\|A^*\| = \|A\|$ and

$$\|A\|^2 = \|A^*A\|.$$

8. Define $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(Tf)(x) = \int_{-\infty}^{x} e^{-(x-y)} f(y) \, dy.$$

Show that T is well-defined and bounded.
(b) Show that

$$\lambda = \frac{1}{1 + i\omega}$$

is in the continuous spectrum of T for every $\omega \in \mathbb{R}$.