Sample Prelim Problems
Integration
Fall 2012

1. (Spring, 2012) For $u \in L^1(0,\infty)$, consider the integral

$$v(x) = \int_0^\infty \frac{u(y)}{x+y}\, dy$$

defined for $x > 0$. Show that $v(x)$ is infinitely differentiable away from the origin. Prove that $v' \in L^1(\epsilon, \infty)$ for any $\epsilon > 0$. How does $v(x)$ behave as $x \to 0^+$?

2. Show that, for every $\alpha \in (0;1]$, there is a constant C_α such that

$$\int_0^\infty \frac{e^{-\lambda t}}{\sqrt{1+t^2}}\, dt \leq \frac{C_\alpha}{\lambda^\alpha}$$

for all $\lambda > 0$, but the analogous inequality for $\alpha = 0$ is false.

3. Show, with justification, that

$$\int_0^\infty \frac{x}{e^x - 1}\, dx = \sum_{n=1}^\infty \frac{1}{n^2}.$$

4. (Fall, 2009) For $\epsilon > 0$, let η_ϵ denote the family of standard mollifiers on $\mathbb{R}^2$. Given $u \in L^2(\mathbb{R}^2)$, define the functions

$$u_\epsilon = \eta_\epsilon * u.$$

Prove that

$$\epsilon \|Du\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{L^2(\mathbb{R}^2)}$$

where the constant C depends on the mollifying function, but not on u.

5. Suppose that $f_n, f \in L^p(\mathbb{R})$, where $1 \leq p < \infty$, $f_n \to f$ pointwise a.e., and $\|f_n\|_{L^p} \to \|f\|_{L^p}$ as $n \to \infty$. Show that $\|f - f_n\|_{L^p} \to 0$. Does this result remain true for $p = \infty$? Hint. Note that

$$|a - b|^p \leq 2^{p-1} \left(|a|^p + |b|^p\right).$$