Sample Prelim Problems
Distributions and Sobolev Spaces
Fall 2012

1. (Spring, 2012) Consider on $\mathbb{R}^2$ the distribution defined by the function

$$E(x,t) = \begin{cases} 1/2 & \text{if } t - |x| > 0, \\ 0 & \text{if } t - |x| < 0. \end{cases}$$

Compute the distributional derivative

$$\frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 E}{\partial x^2}.$$

2. Compute the distributional Fourier transform of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

3. Let

$$K_n(x) = \frac{\sin nx}{\pi x}$$

Show that $K_n \rightharpoonup \delta$ in $\mathcal{D}'(\mathbb{R})$ as $n \to \infty$.

4. Find all distributions $T \in \mathcal{D}'(\mathbb{R})$ that satisfy

$$x^{2012} T = 0.$$

5. Show that the PDE on $\mathbb{R}^3$

$$\frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 u}{\partial z^2} + 3u = f$$

has a unique solution $u \in \mathcal{S}'(\mathbb{R}^n)$ for every $f \in \mathcal{S}'(\mathbb{R}^n)$. For what $m \in \mathbb{Z}$ is $u \in H^m(\mathbb{R}^3)$ when $f \in L^2(\mathbb{R}^n)$?

6. (a) (Fall, 2009) Let $B \subset \mathbb{R}^3$ denote the unit ball $\{|x| < 1\}$. Prove that $\log |x| \in H^1(B)$.

(b) (Spring, 2011 *and* Fall, 2011) Let $u(x) = (1 + \log |x|)^{-1}$ Show that $u \in W^{1,1}(\Omega)$ and that $v(0) = 0$, but that $u/x \notin L^1(\Omega)$.

7. Let $1 \leq p < n$. The Sobolev conjugate $p < p^* < \infty$ is given by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Suppose that $u \in C_c^\infty(\mathbb{R}^n)$ is a non-zero, smooth function with compact support. For $\epsilon > 0$ define $u^\epsilon \in C_c^\infty(\mathbb{R}^n)$ by

$$u^\epsilon(x) = u\left(\frac{x}{\epsilon}\right).$$

(a) Compute the norms $\|u^\epsilon\|_{L^p}$, $\|Du^\epsilon\|_{L^p}$, and $\|u^\epsilon\|_{L^{p^*}}$ in terms of the corresponding norms of u.

(b) According to the Sobolev embedding theorem $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ and this embedding is continuous. Show that it is not compact.

8. Suppose that $u \in W^{1,1}(0,1)$ with weak derivative $Du \in L^1(0,1)$.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with bounded derivative, show that $f \circ u : (0,1) \to \mathbb{R}$ is weakly differentiable with weak derivative $f'(u)Du$. Hint: Approximate u by a sequence (u_n) of smooth functions such that $u_n \to u$ in $W^{1,1}(0,1)$ and pointwise a.e.

(b) Let $u^+(x) = \max\{u(x), 0\}$ denote the positive part of u. Show that u^+ is weakly differentiable with weak derivative

$$Du^+(x) = \begin{cases} Du(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) \leq 0. \end{cases}$$

Hint: Approximate u^+ by $f_\epsilon(u)$ where

$$f_\epsilon(u) = \begin{cases} (u^2 + \epsilon^2)^{1/2} - \epsilon & \text{if } u > 0, \\ 0 & \text{if } u \leq 0. \end{cases}$$